Exercise 10

Solve the wave equation with Robin boundary conditions under the assumption that (18) holds.

Solution

The PDE to solve here is

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < l,$$

subject to the Robin boundary conditions

$$\begin{aligned} u_x(0,t) - a_0 u(0,t) &= 0 \\ u_x(l,t) + a_l u(l,t) &= 0 \end{aligned}$$

with $a_0 < 0$, $a_l > -a_0$, and

$$a_0 + a_l < -a_0 a_l l. \tag{18}$$

Since the PDE and boundary conditions are linear and homogeneous, the method of separation of variables can be applied. Assume a product solution of the form $u(x,t) = X(x)T(t)$ and plug it into the PDE

$$u_{tt} = c^2 u_{xx} \quad \rightarrow \quad XT'' = c^2 X'' T$$

and the boundary conditions.

$$\begin{aligned} u_x(0,t) - a_0 u(0,t) = 0 \quad &\rightarrow \quad X'(0)T(t) - a_0 X(0)T(t) = 0 \quad \rightarrow \quad X'(0) - a_0 X(0) = 0 \\ u_x(l,t) + a_l u(l,t) = 0 \quad &\rightarrow \quad X'(l)T(t) + a_l X(l)T(t) = 0 \quad \rightarrow \quad X'(l) + a_l X(l) = 0 \end{aligned}$$

Now separate variables in the PDE: bring all functions of t and constants to the left side and all functions of x to the right side. The final answer would be the same if all constants were brought to the right side.

$$\frac{T''}{c^2 T} = \frac{X''}{X}$$

Because we have a function of t equal to a function of x for all t and x, both must be equal to a constant.

$$\frac{T''}{c^2 T} = \frac{X''}{X} = k$$

Values of this constant and the functions $X(x)$ associated with them for which the boundary conditions are satisfied are known as eigenvalues and eigenfunctions, respectively.

Determination of Positive Eigenvalues: $k = \beta^2$

Assuming k is positive, the ODE for X becomes

$$X'' = \beta^2 X.$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \beta x + C_2 \sinh \beta x$$

Apply the boundary conditions now to determine C_1 and C_2.

$$\begin{aligned} X'(0) - a_0 X(0) &= -a_0 C_1 + \beta C_2 = 0 \\ X'(l) + a_l X(l) &= \beta(C_1 \sinh \beta l + C_2 \cosh \beta l) + a_l(C_1 \cosh \beta l + C_2 \sinh \beta l) = 0 \end{aligned}$$

Solve the first equation for C_2 and factor the left side of the second equation.

$$C_2 = \frac{a_0}{\beta}C_1$$
$$C_1(\beta \sinh \beta l + a_l \cosh \beta l) + C_2(\beta \cosh \beta l + a_l \sinh \beta l) = 0$$

Substitute the expression for C_2 into the second equation.

$$C_1(\beta \sinh \beta l + a_l \cosh \beta l) + \frac{a_0}{\beta}C_1(\beta \cosh \beta l + a_l \sinh \beta l) = 0$$

Divide both sides by C_1 and factor the left side.

$$\left(\beta + \frac{a_0 a_l}{\beta}\right)\sinh \beta l + (a_l + a_0)\cosh \beta l = 0$$

We find that β satisfies a transcendental equation.

$$\tanh \beta l = -\frac{a_l + a_0}{\beta + \frac{a_0 a_l}{\beta}}$$
$$\tanh \beta l = -\frac{(a_0 + a_l)\beta}{\beta^2 + a_0 a_l} \tag{1}$$

The solutions of this equation are where the intersections of the graphs, $y = \tanh \beta l$ and $y = -[(a_0 + a_l)\beta]/(\beta^2 + a_0 a_l)$, occur. Since these are both odd functions of β and $k = \beta^2$, negative values of β yield redundant values of k. The point is that only positive values of β need to be considered. If a_0 and a_l are chosen such that $a_0 < 0$, $a_l > -a_0$, and $a_0 + a_l < -a_0 a_l l$, then the graphs will look similar to those in Figure 1 and have one intersection.

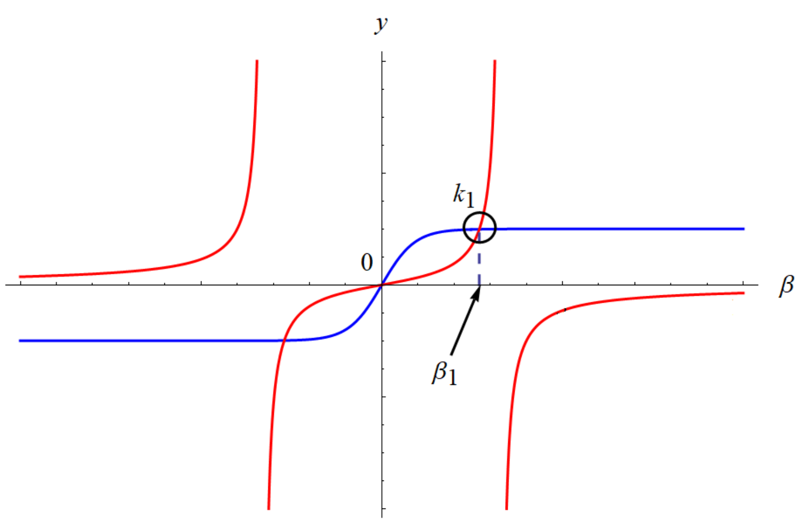

Figure 1: This is a plot of $y = \tanh \beta l$ in blue and $y = -[(a_0 + a_l)\beta]/(\beta^2 + a_0 a_l)$ in red. Since $a_0 < 0$, $a_l > -a_0$, and $a_0 + a_l < -a_0 a_l l$, there is exactly one intersection.

Let β_1 denote the positive solution to the transcendental equation in equation (1). The eigenfunction associated with it is

$$X(x) = C_1 \cosh \beta x + \frac{a_0}{\beta}C_1 \sinh \beta x$$
$$= C_1\left(\cosh \beta x + \frac{a_0}{\beta}\sinh \beta x\right).$$

Thus,

$$X(x) = \cosh \beta_1 x + \frac{a_0}{\beta_1} \sinh \beta_1 x.$$

Now we will solve the ODE for $T(t)$.

$$\frac{T''}{c^2 T} = \beta^2$$

Multiply both sides by $c^2 T$.

$$T'' = c^2 \beta^2 T$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$T(t) = C_3 \cosh c\beta t + C_4 \sinh c\beta t$$

Determination of the Zero Eigenvalue: $k = 0$

Assuming $k = 0$, the ODE for X simplifies to

$$X'' = 0.$$

Integrate both sides with respect to x.

$$X' = C_5$$

Integrate both sides with respect to x once more to solve for X.

$$X(x) = C_5 x + C_6$$

Apply the boundary conditions now to determine C_5 and C_6.

$$\begin{aligned} X'(0) - a_0 X(0) &= C_5 - a_0 C_6 = 0 \\ X'(l) + a_l X(l) &= C_5 + a_l (C_5 l + C_6) = 0 \end{aligned}$$

Solve the first equation for C_5 and factor the left side of the second equation.

$$\begin{aligned} &C_5 = a_0 C_6 \\ &C_5 (1 + a_l l) + C_6 a_l = 0 \end{aligned}$$

Substitute the expression for C_5 into the second equation.

$$a_0 C_6 (1 + a_l l) + C_6 a_l = 0$$

Divide both sides by C_6, expand the left side, and bring $a_0 a_l l$ to the right side.

$$a_0 + a_l = -a_0 a_l l$$

Zero is not an eigenvalue because one of our assumptions is that $a_0 + a_l < -a_0 a_l l$.

Determination of Negative Eigenvalues: $k = -\gamma^2$

Assuming k is negative, the ODE for X becomes

$$X'' = -\gamma^2 X.$$

The general solution can be written in terms of sine and cosine.

$$X(x) = C_7 \cos\gamma x + C_8 \sin\gamma x$$

Apply the boundary conditions now to determine C_7 and C_8.

$$\begin{aligned} X'(0) - a_0 X(0) &= -a_0 C_7 + \gamma C_8 = 0 \\ X'(l) + a_l X(l) &= \gamma(-C_7 \sin\gamma l + C_8 \cos\gamma l) + a_l(C_7 \cos\gamma l + C_8 \sin\gamma l) = 0 \end{aligned}$$

Solve the first equation for C_8 and factor the left side of the second equation.

$$\begin{aligned} C_8 &= \frac{a_0}{\gamma} C_7 \\ C_7(-\gamma \sin\gamma l + a_l \cos\gamma l) &+ C_8(\gamma \cos\gamma l + a_l \sin\gamma l) = 0 \end{aligned}$$

Substitute the expression for C_8 into the second equation.

$$C_7(-\gamma \sin\gamma l + a_l \cos\gamma l) + \frac{a_0}{\gamma} C_7(\gamma \cos\gamma l + a_l \sin\gamma l) = 0$$

Divide both sides by C_7 and factor the left side.

$$\left(-\gamma + \frac{a_0 a_l}{\gamma}\right) \sin\gamma l + (a_l + a_0) \cos\gamma l = 0$$

We find that γ satisfies a transcendental equation.

$$\begin{aligned} \tan\gamma l &= -\frac{a_l + a_0}{-\gamma + \frac{a_0 a_l}{\gamma}} \\ \tan\gamma l &= \frac{(a_0 + a_l)\gamma}{\gamma^2 - a_0 a_l} \qquad (2) \end{aligned}$$

The solutions of this equation are where the intersections of the graphs, $y = \tan\gamma l$ and $y = [(a_0 + a_l)\gamma]/(\gamma^2 - a_0 a_l)$, occur. Since these are both odd functions of γ and $k = -\gamma^2$, negative values of γ yield redundant values of k. The point is that only positive values of γ need to be considered. If a_0 and a_l are chosen such that $a_0 < 0$, $a_l > -a_0$, and $a_0 + a_l < -a_0 a_l l$, then the graphs will look similar to those in Figure 2 and have an infinite number of intersections.

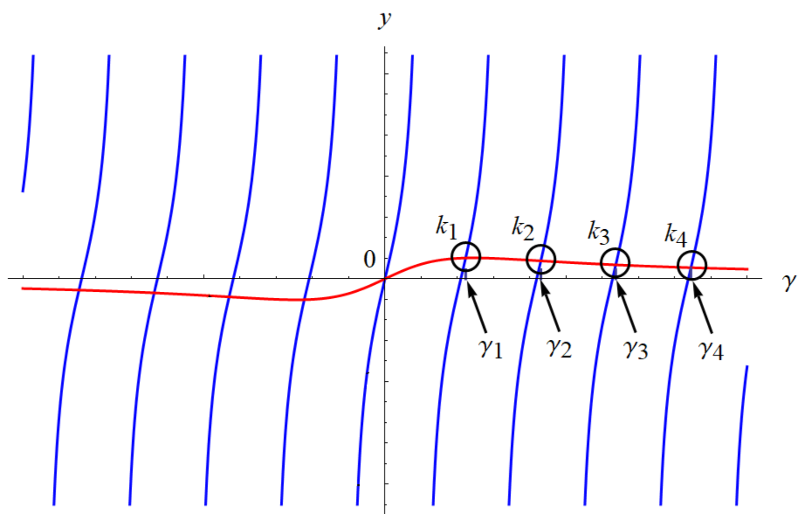

Figure 2: This is a plot of $y = \tan \gamma l$ in blue and $y = [(a_0 + a_l)\gamma]/(\gamma^2 - a_0 a_l)$ in red. Since $a_0 < 0$, $a_l > -a_0$, and $a_0 + a_l < -a_0 a_l l$, there are infinitely many intersections.

Let γ_n denote the positive solutions to the transcendental equation in equation (2). The eigenfunctions associated with them are

$$\begin{aligned} X(x) &= C_7 \cos \gamma x + C_8 \sin \gamma x \\ &= C_7 \cos \gamma x + \frac{a_0}{\gamma} C_7 \sin \gamma x \\ &= C_7 \left(\cos \gamma x + \frac{a_0}{\gamma} \sin \gamma x \right). \end{aligned}$$

Thus,

$$X_n(x) = \cos \gamma_n x + \frac{a_0}{\gamma_n} \sin \gamma_n x.$$

Now we will solve the ODE for $T(t)$.

$$\frac{T''}{c^2 T} = -\gamma^2$$

Multiply both sides by $c^2 T$.

$$T'' = -c^2 \gamma^2 T$$

The general solution can be written in terms of sine and cosine.

$$T(t) = C_9 \cos c\gamma t + C_{10} \sin c\gamma t$$

According to the principle of linear superposition, the general solution for $u(x, t)$ is a linear combination of $X(x)T(t)$ over all the eigenvalues. Therefore,

$$\begin{aligned} u(x,t) = (A \cosh c\beta_1 t + B \sinh c\beta_1 t) \left(\cosh \beta_1 x + \frac{a_0}{\beta_1} \sinh \beta_1 x \right) \\ + \sum_{n=1}^{\infty} (D_n \cos c\gamma_n t + E_n \sin c\gamma_n t) \left(\cos \gamma_n x + \frac{a_0}{\gamma_n} \sin \gamma_n x \right). \end{aligned}$$

If we had two initial conditions, we could determine the coefficients, A, B, D_n, and E_n. Also, note that the hyperbolic trigonometric functions can be written in terms of exponential functions.