Exercise 11

(a) Prove that the (total) energy is conserved for the wave equation with Dirichlet BCs, where the energy is defined to be

$$E = \frac{1}{2}\int_0^l (c^{-2}u_t^2 + u_x^2)\,dx.$$

(Compare this definition with Section 2.2.)

(b) Do the same for the Neumann BCs.

(c) For the Robin BCs, show that

$$E_R = \frac{1}{2}\int_0^l (c^{-2}u_t^2 + u_x^2)\,dx + \frac{1}{2}a_l[u(l,t)]^2 + \frac{1}{2}a_0[u(0,t)]^2$$

is conserved. Thus, while the total energy E_R is still a constant, some of the internal energy is "lost" to the boundary if a_0 and a_l are positive and "gained" from the boundary if a_0 and a_l are negative.