Exercise 12

Consider the unusual eigenvalue problem

$$-v_{xx} = \lambda v \qquad \text{for } 0 < x < l$$
$$v_x(0) = v_x(l) = \frac{v(l) - v(0)}{l}.$$

(a) Show that $\lambda = 0$ is a double eigenvalue.

(b) Get an equation for the positive eigenvalues $\lambda > 0$.

(c) Letting $\gamma = \frac{1}{2} l\sqrt{\lambda}$, reduce the equation in part (b) to the equation

$$\gamma \sin\gamma \cos\gamma = \sin^2\gamma.$$

(d) Use part (c) to find half of the eigenvalues explicitly and half of them graphically.

(e) Assuming that all the eigenvalues are nonnegative, make a list of all the eigenfunctions.

(f) Solve the problem $u_t = ku_{xx}$ for $0 < x < l$, with the BCs given above, and with $u(x, 0) = \phi(x)$.

(g) Show that, as $t \to \infty$, $\lim u(x, t) = A + Bx$ for some constants A, B, assuming that you can take limits term by term.