Exercise 15

Find the equation for the eigenvalues λ of the problem

$$(\kappa(x)X')' + \lambda\rho(x)X = 0 \qquad \text{for } 0 < x < l \text{ with } X(0) = X(l) = 0,$$

where $\kappa(x) = \kappa_1^2$ for $x < a$, $\kappa(x) = \kappa_2^2$ for $x > a$, $\rho(x) = \rho_1^2$ for $x < a$, and $\rho(x) = \rho_2^2$ for $x > a$. All these constants are positive and $0 < a < l$.

Solution

With $\kappa(x)$ and $\rho(x)$ defined as they are, the ODE above is equivalent to the following system.

$$\begin{cases} (\kappa_1^2 X')' + \lambda\rho_1^2 X = 0 & x < a \\ (\kappa_2^2 X')' + \lambda\rho_2^2 X = 0 & x > a \end{cases}$$

$$\begin{cases} \kappa_1^2 X'' + \lambda\rho_1^2 X = 0 & x < a \\ \kappa_2^2 X'' + \lambda\rho_2^2 X = 0 & x > a \end{cases}$$

$$\begin{cases} X'' = -\lambda\dfrac{\rho_1^2}{\kappa_1^2}X & x < a \\ X'' = -\lambda\dfrac{\rho_2^2}{\kappa_2^2}X & x > a \end{cases}$$

The solution to the ODE depends on whether λ is positive, zero, or negative.

Determination of Positive Eigenvalues: $\lambda = \mu^2$

Assuming λ is positive, the system of differential equations for X becomes

$$\begin{cases} X'' = -\mu^2\dfrac{\rho_1^2}{\kappa_1^2}X & x < a \\ X'' = -\mu^2\dfrac{\rho_2^2}{\kappa_2^2}X & x > a \end{cases}.$$

The solution to the system can be written in terms of sine and cosine.

$$X(x) = \begin{cases} C_1 \cos\mu\frac{\rho_1}{\kappa_1}x + C_2 \sin\mu\frac{\rho_1}{\kappa_1}x & x < a \\ C_3 \cos\mu\frac{\rho_2}{\kappa_2}x + C_4 \sin\mu\frac{\rho_2}{\kappa_2}x & x > a \end{cases}$$

The boundary condition, $X(0) = 0$, applies to the first case, and the boundary condition, $X(l) = 0$, applies to the second case.

$$X(0) = C_1 = 0$$

$$X(l) = C_3 \cos\mu\frac{\rho_2}{\kappa_2}l + C_4 \sin\mu\frac{\rho_2}{\kappa_2}l \quad \rightarrow \quad C_3 = -C_4\frac{\sin\mu\frac{\rho_2}{\kappa_2}l}{\cos\mu\frac{\rho_2}{\kappa_2}l}$$

So the solution becomes

$$X(x) = \begin{cases} C_2 \sin \mu \frac{\rho_1}{\kappa_1} x & x < a \\ -C_4 \frac{\sin \mu \frac{\rho_2}{\kappa_2} l}{\cos \mu \frac{\rho_2}{\kappa_2} l} \cos \mu \frac{\rho_2}{\kappa_2} x + C_4 \sin \mu \frac{\rho_2}{\kappa_2} x & x > a \end{cases}$$

$$= \begin{cases} C_2 \sin \mu \frac{\rho_1}{\kappa_1} x & x < a \\ C_4 \left(\sin \mu \frac{\rho_2}{\kappa_2} x - \frac{\sin \mu \frac{\rho_2}{\kappa_2} l}{\cos \mu \frac{\rho_2}{\kappa_2} l} \cos \mu \frac{\rho_2}{\kappa_2} x \right) & x > a \end{cases}.$$

Replace C_4 with a new integration constant, $C_5 \cos \mu \frac{\rho_2}{\kappa_2} l$, to simplify the second expression.

$$= \begin{cases} C_2 \sin \mu \frac{\rho_1}{\kappa_1} x & x < a \\ C_5 \left(\sin \mu \frac{\rho_2}{\kappa_2} x \cos \mu \frac{\rho_2}{\kappa_2} l - \sin \mu \frac{\rho_2}{\kappa_2} l \cos \mu \frac{\rho_2}{\kappa_2} x \right) & x > a \end{cases}$$

The expression in parentheses is the angle subtraction formula for sine.

$$= \begin{cases} C_2 \sin \mu \frac{\rho_1}{\kappa_1} x & x < a \\ C_5 \sin \mu \frac{\rho_2}{\kappa_2} (x - l) & x > a \end{cases}$$

To figure out the two remaining constants, we require the eigenfunction and its slope to be continuous at $x = a$. That is,

$$X(a-) = X(a+) \quad \rightarrow \quad C_2 \sin \mu \frac{\rho_1}{\kappa_1} a = C_5 \sin \mu \frac{\rho_2}{\kappa_2} (a - l)$$

$$X'(a-) = X'(a+) \quad \rightarrow \quad C_2 \cancel{\mu} \frac{\rho_1}{\kappa_1} \cos \mu \frac{\rho_1}{\kappa_1} a = C_5 \cancel{\mu} \frac{\rho_2}{\kappa_2} \cos \mu \frac{\rho_2}{\kappa_2} (a - l)$$

Divide the first equation by the second one to obtain the equation for μ.

$$\frac{\kappa_1}{\rho_1} \tan \mu \frac{\rho_1}{\kappa_1} a = \frac{\kappa_2}{\rho_2} \tan \mu \frac{\rho_2}{\kappa_2} (a - l)$$

This is a transcendental equation, so we leave it as it is. To get the answer at the back of the book, factor out a minus sign from the argument of tangent,

$$\frac{\kappa_1}{\rho_1} \tan \mu \frac{\rho_1}{\kappa_1} a = -\frac{\kappa_2}{\rho_2} \tan \mu \frac{\rho_2}{\kappa_2} (l - a),$$

invert both sides,

$$\frac{\rho_1}{\kappa_1} \cot \mu \frac{\rho_1}{\kappa_1} a = -\frac{\rho_2}{\kappa_2} \cot \mu \frac{\rho_2}{\kappa_2} (l - a),$$

and then bring all terms to the left side.

$$\frac{\rho_1}{\kappa_1} \cot \mu \frac{\rho_1}{\kappa_1} a + \frac{\rho_2}{\kappa_2} \cot \mu \frac{\rho_2}{\kappa_2} (l - a) = 0 \qquad (1)$$

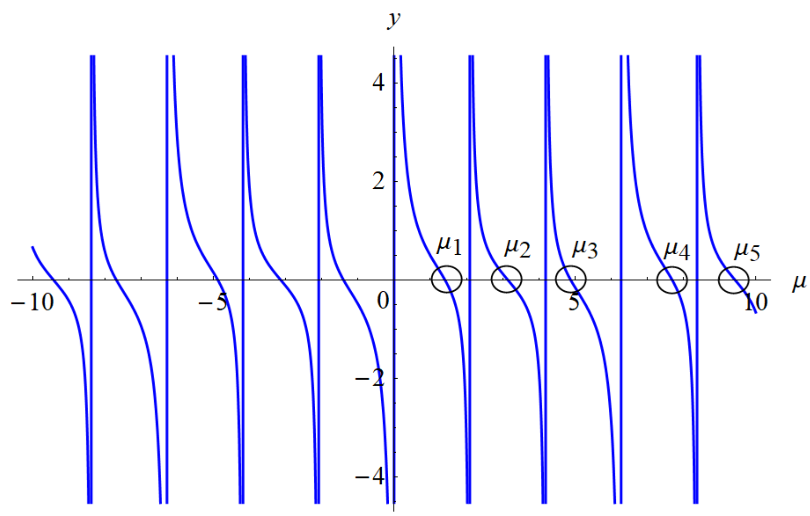

Figure 1: This is a plot of the function on the left side of equation (1) with $\rho_1/\kappa_1 = 1/2$, $a = 3$, $\rho_2/\kappa_2 = 1/4$, and $l = 5$. The values of μ that solve the equation are the zeroes of the function. Since the eigenvalues are given by $\lambda = \mu^2$ and cotangent is an odd function, negative values of μ give redundant values for λ.

Determination of the Zero Eigenvalue: $\lambda = 0$

Assuming λ is zero, the system of differential equations for X becomes

$$\begin{cases} X'' = 0 & x < a \\ X'' = 0 & x > a \end{cases}.$$

The solution to the system is a linear function.

$$X(x) = \begin{cases} C_6x + C_7 & x < a \\ C_8x + C_9 & x > a \end{cases}$$

The boundary condition, $X(0) = 0$, applies to the first case, and the boundary condition, $X(l) = 0$, applies to the second case.

$$\begin{aligned} X(0) &= C_7 = 0 \\ X(l) &= C_8l + C_9 = 0 \quad \rightarrow \quad C_9 = -C_8l \end{aligned}$$

So the solution becomes

$$\begin{aligned} X(x) &= \begin{cases} C_6x & x < a \\ C_8x - C_8l & x > a \end{cases} \\ &= \begin{cases} C_6x & x < a \\ C_8(x-l) & x > a \end{cases}. \end{aligned}$$

To figure out the two remaining constants, we require the eigenfunction and its slope to be continuous at $x = a$. That is,

$$\begin{aligned} X(a-) = X(a+) \quad &\rightarrow \quad C_6a = C_8(a-l) \\ X'(a-) = X'(a+) \quad &\rightarrow \quad C_6 = C_8. \end{aligned}$$

These two equations imply that $C_6 = 0$ and $C_8 = 0$, which means only the trivial solution is obtained from considering $\lambda = 0$. Hence, zero is not an eigenvalue.

Determination of Negative Eigenvalues: $\lambda = -\eta^2$

Assuming λ is negative, the system of differential equations for X becomes

$$\begin{cases} X'' = \eta^2 \dfrac{\rho_1^2}{\kappa_1^2} X & x < a \\ X'' = \eta^2 \dfrac{\rho_2^2}{\kappa_2^2} X & x > a \end{cases}.$$

The solution to the system can be written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = \begin{cases} C_{10} \cosh \eta \frac{\rho_1}{\kappa_1} x + C_{11} \sinh \eta \frac{\rho_1}{\kappa_1} x & x < a \\ C_{12} \cosh \eta \frac{\rho_2}{\kappa_2} x + C_{13} \sinh \eta \frac{\rho_2}{\kappa_2} x & x > a \end{cases}$$

The boundary condition, $X(0) = 0$, applies to the first case, and the boundary condition, $X(l) = 0$, applies to the second case.

$$X(0) = C_{10} = 0$$

$$X(l) = C_{12} \cosh \eta \frac{\rho_2}{\kappa_2} l + C_{13} \sinh \eta \frac{\rho_2}{\kappa_2} l \quad \rightarrow \quad C_{12} = -C_{13} \frac{\sinh \eta \frac{\rho_2}{\kappa_2} l}{\cosh \eta \frac{\rho_2}{\kappa_2} l}$$

So the solution becomes

$$X(x) = \begin{cases} C_{11} \sinh \eta \frac{\rho_1}{\kappa_1} x & x < a \\ -C_{13} \frac{\sinh \eta \frac{\rho_2}{\kappa_2} l}{\cosh \eta \frac{\rho_2}{\kappa_2} l} \cosh \eta \frac{\rho_2}{\kappa_2} x + C_{13} \sinh \eta \frac{\rho_2}{\kappa_2} x & x > a \end{cases}$$

$$= \begin{cases} C_{11} \sinh \eta \frac{\rho_1}{\kappa_1} x & x < a \\ C_{13} \left(\sinh \eta \frac{\rho_2}{\kappa_2} x - \frac{\sinh \eta \frac{\rho_2}{\kappa_2} l}{\cosh \eta \frac{\rho_2}{\kappa_2} l} \cosh \eta \frac{\rho_2}{\kappa_2} x \right) & x > a \end{cases}.$$

Replace C_{13} with a new integration constant, $C_{14} \cosh \eta \frac{\rho_2}{\kappa_2} l$, to simplify the second expression.

$$= \begin{cases} C_{11} \sinh \eta \frac{\rho_1}{\kappa_1} x & x < a \\ C_{14} \left(\sinh \eta \frac{\rho_2}{\kappa_2} x \cosh \eta \frac{\rho_2}{\kappa_2} l - \sinh \eta \frac{\rho_2}{\kappa_2} l \cosh \eta \frac{\rho_2}{\kappa_2} x \right) & x > a \end{cases}$$

The expression in parentheses is the angle subtraction formula for hyperbolic sine.

$$= \begin{cases} C_{11} \sinh \eta \frac{\rho_1}{\kappa_1} x & x < a \\ C_{14} \sinh \eta \frac{\rho_2}{\kappa_2} (x - l) & x > a \end{cases}$$

To figure out the two remaining constants, we require the eigenfunction and its slope to be continuous at $x = a$. That is,

$$X(a-) = X(a+) \quad \rightarrow \quad C_{11} \sinh \eta \frac{\rho_1}{\kappa_1} a = C_{14} \sinh \eta \frac{\rho_2}{\kappa_2} (a - l)$$

$$X'(a-) = X'(a+) \quad \rightarrow \quad C_{11} \cancel{\eta} \frac{\rho_1}{\kappa_1} \cosh \eta \frac{\rho_1}{\kappa_1} a = C_{14} \cancel{\eta} \frac{\rho_2}{\kappa_2} \cosh \eta \frac{\rho_2}{\kappa_2} (a - l).$$

Divide the first equation by the second one to obtain the equation for η.

$$\frac{\kappa_1}{\rho_1}\tanh\eta\frac{\rho_1}{\kappa_1}a = \frac{\kappa_2}{\rho_2}\tanh\eta\frac{\rho_2}{\kappa_2}(a-l)$$

Factor a minus sign from the argument of hyperbolic tangent,

$$\frac{\kappa_1}{\rho_1}\tanh\eta\frac{\rho_1}{\kappa_1}a = -\frac{\kappa_2}{\rho_2}\tanh\eta\frac{\rho_2}{\kappa_2}(l-a),$$

invert both sides,

$$\frac{\rho_1}{\kappa_1}\coth\eta\frac{\rho_1}{\kappa_1}a = -\frac{\rho_2}{\kappa_2}\coth\eta\frac{\rho_2}{\kappa_2}(l-a),$$

and bring both terms to the left side.

$$\frac{\rho_1}{\kappa_1}\coth\eta\frac{\rho_1}{\kappa_1}a + \frac{\rho_2}{\kappa_2}\coth\eta\frac{\rho_2}{\kappa_2}(l-a) = 0 \tag{2}$$

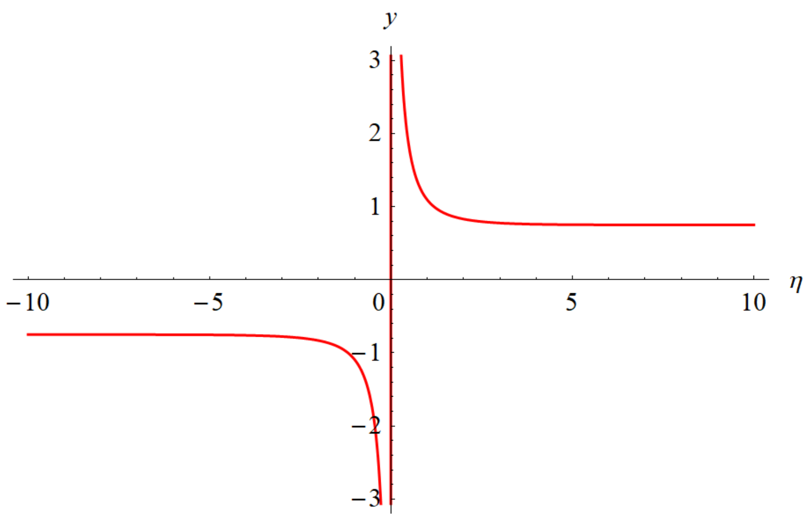

Figure 2: This is a plot of the function on the left side of equation (2) with $\rho_1/\kappa_1 = 1/2$, $a = 3$, $\rho_2/\kappa_2 = 1/4$, and $l = 5$. The values of η that solve the equation are the zeroes of the function. Since there are none, there are no negative eigenvalues.

Therefore, the eigenvalues are $\lambda = \mu^2$, where μ is obtained from the equation,

$$\frac{\rho_1}{\kappa_1}\cot\mu\frac{\rho_1}{\kappa_1}a + \frac{\rho_2}{\kappa_2}\cot\mu\frac{\rho_2}{\kappa_2}(l-a) = 0.$$