# Exercise 2

Consider the eigenvalue problem with Robin BCs at both ends:

$$-X'' = \lambda X$$
$$X'(0) - a_0 X(0) = 0, \quad X'(l) + a_l X(l) = 0.$$

(a) Show that $\lambda = 0$ is an eigenvalue if and only if $a_0 + a_l = -a_0 a_l l$.

(b) Find the eigenfunctions corresponding to the zero eigenvalue. (*Hint:* First solve the ODE for $X(x)$. The solutions are not sines or cosines.)

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**Solution**

**Part (a)**

If $\lambda = 0$, then the ODE simplifies to

$$X'' = 0.$$

The general solution for $X$ is a linear function.

$$X(x) = C_1 x + C_2$$

Apply the two provided boundary conditions to determine $C_1$ and $C_2$.

$$X'(0) - a_0 X(0) = C_1 - a_0(C_2) = 0 \quad (1)$$
$$X'(l) + a_l X(l) = C_1 + a_l(C_1 l + C_2) = 0 \quad (2)$$

Solving equation (1) for $C_1$ gives $C_1 = a_0 C_2$. Substitute this into equation (2).

$$a_0 C_2 + a_l(a_0 C_2 l + C_2) = 0$$

Divide both sides by $C_2$.

$$a_0 + a_l(a_0 l + 1) = 0$$

Expand the left side.

$$a_0 + a_l a_0 l + a_l = 0$$

Therefore, zero is an eigenvalue if and only if $a_0 + a_l = -a_0 a_l l$ is satisfied.

**Part (b)**

The eigenfunctions associated with the zero eigenvalue are

$$\begin{aligned} X(x) &= C_1 x + C_2 \\ &= (a_0 C_2)x + C_2 \\ &= C_2(a_0 x + 1). \end{aligned}$$

Therefore,

$$X_0(x) = a_0 x + 1.$$