Exercise 8

Consider again Robin BCs at both ends for arbitrary a_0 and a_l.

(a) In the $a_0 a_l$ plane sketch the hyperbola $a_0 + a_l = -a_0 a_l l$. Indicate the asymptotes. For (a_0, a_l) on this hyperbola, zero is an eigenvalue, according to Exercise 2(a).

(b) Show that the hyperbola separates the whole plane into three regions, depending on whether there are two, one, or no negative eigenvalues.

(c) Label the directions of increasing absorption and radiation on each axis. Label the point corresponding to Neumann BCs.

(d) Where in the plane do the Dirichlet BCs belong?

Solution

Part (a)

In order to sketch the hyperbola, solve the given equation for a_l.

$$a_0 + a_l = -a_0 a_l l$$

$$a_l + a_0 a_l l = -a_0$$

$$a_l(1 + a_0 l) = -a_0$$

$$a_l = -\frac{a_0}{1 + a_0 l}$$

There is a vertical asymptote where the denominator is equal to 0.

$$\text{Vertical Asymptote :} \quad 1 + a_0 l = 0 \quad \rightarrow \quad a_0 = -\frac{1}{l}$$

Also, there is a horizontal asymptote.

$$\begin{aligned} \text{Horizontal Asymptote :} \quad \lim_{a_0 \to \pm\infty} a_l &= \lim_{a_0 \to \pm\infty} -\frac{a_0}{1 + a_0 l} \\ &= \lim_{a_0 \to \pm\infty} -\frac{1}{\frac{1}{a_0} + l} \\ &= -\frac{1}{l} \end{aligned}$$

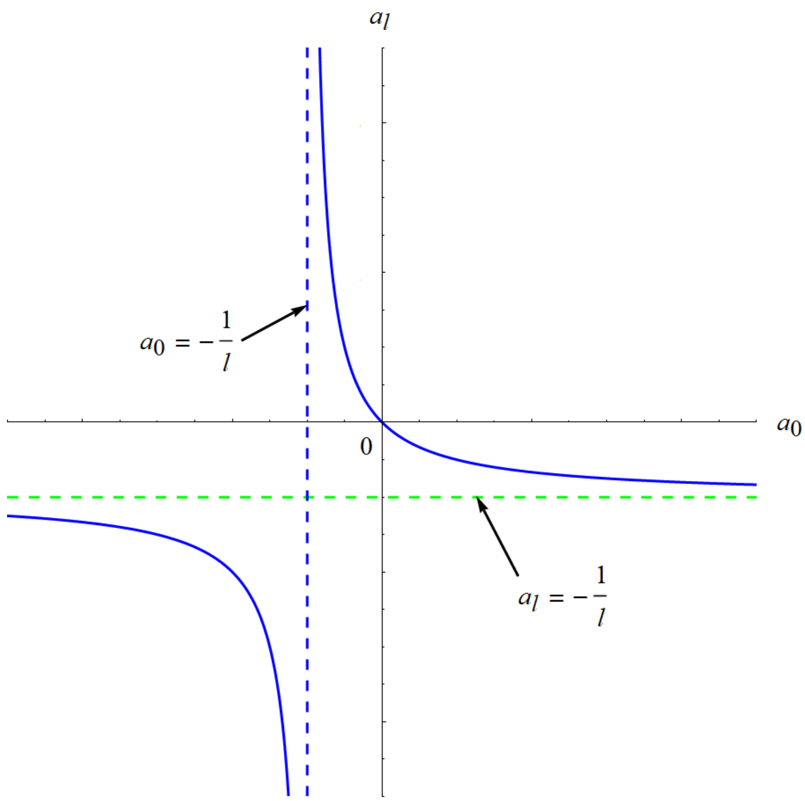

Figure 1: This is a plot of the hyperbola $a_0 + a_l = -a_0 a_l l$ in the a_0a_l-plane. The vertical (blue) and horizontal (green) asymptotes are included as well.

Part (b)

The eigenvalue problem under consideration is

$$-X'' = \lambda X$$

subject to the Robin boundary conditions,

$$\begin{aligned} X'(0) - a_0 X(0) &= 0 \\ X'(l) + a_l X(l) &= 0. \end{aligned}$$

If we want to find the negative eigenvalues, then we set $\lambda = -\gamma^2$. Solving the differential equation and applying the boundary conditions yields an equation for γ. This is equation (16) in the textbook.

$$\tanh \gamma l = -\frac{(a_0 + a_l)\gamma}{\gamma^2 + a_0 a_l}. \tag{16}$$

Intersections of the graphs of $y = \tanh \gamma l$ and $y = -[(a_0 + a_l)\gamma]/(\gamma^2 + a_0 a_l)$ give the eigenvalues we're looking for. Since these functions of γ are odd and $\lambda = -\gamma^2$, intersections that occur at negative values of γ yield redundant values of λ. The point is that we only need to consider intersections at positive values of γ. The nature of the rational function changes, depending on what a_0 and a_l are. One by one we will go through each region of the a_0a_l-plane to determine the number of intersections it has with the hyperbolic tangent function.

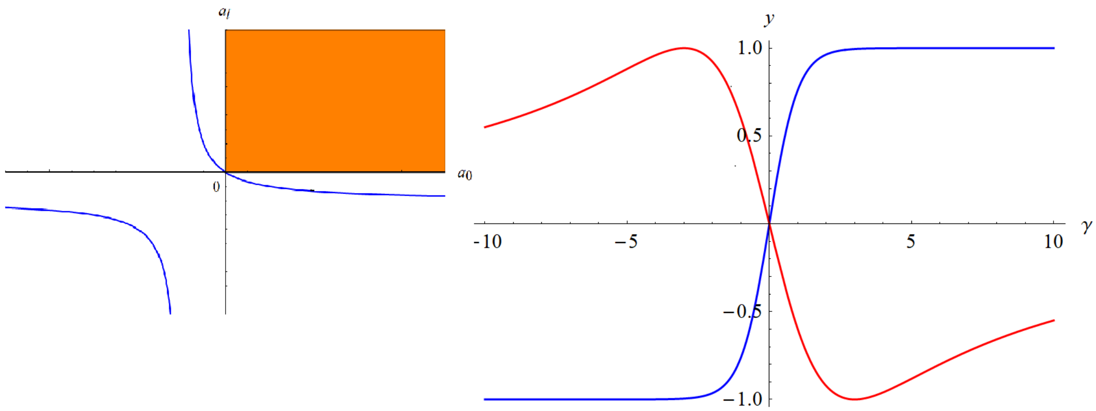

Figure 2: Choosing any two values for a_0 and a_l in the shaded region of the a_0a_l-plane gives us the two plots on the right. The shaded region is $a_0 > 0$ and $a_l > 0$, which means physically that there is radiation at both boundaries. In blue is a plot of $y = \tanh \gamma l$ and in red is a plot of $y = -[(a_0+a_l)\gamma]/(\gamma^2+a_0a_l)$. Since l is positive, there are no intersections and hence no eigenvalues in this region.

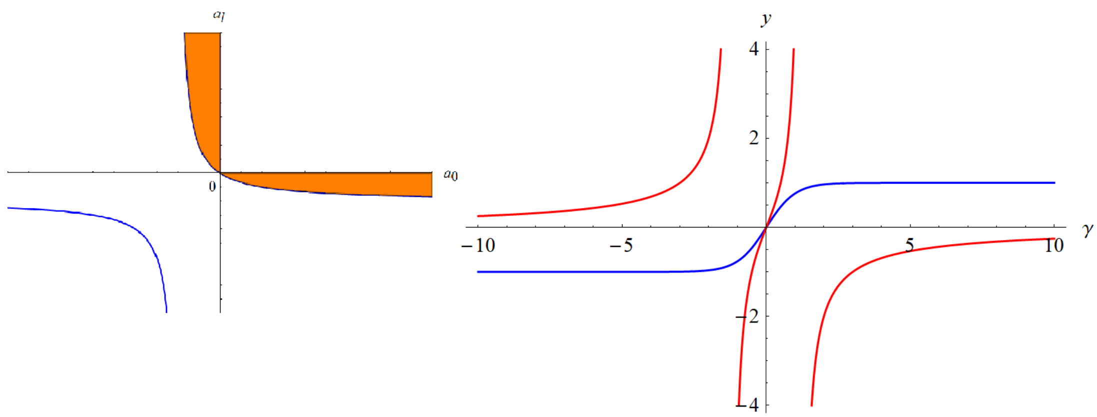

Figure 3: Choosing any two values for a_0 and a_l in the shaded region of the a_0a_l-plane gives us the two plots on the right. The shaded region is the union of $a_0 < 0$ and $a_l > 0$ and $a_0 + a_l > -a_0a_l l$ as well as $a_0 > 0$ and $a_l < 0$ and $a_0 + a_l > -a_0a_l l$, which means physically that there is more radiation than absorption at the boundaries. In blue is a plot of $y = \tanh \gamma l$ and in red is a plot of $y = -[(a_0 + a_l)\gamma]/(\gamma^2 + a_0a_l)$. There are no intersections and hence no eigenvalues in this region.

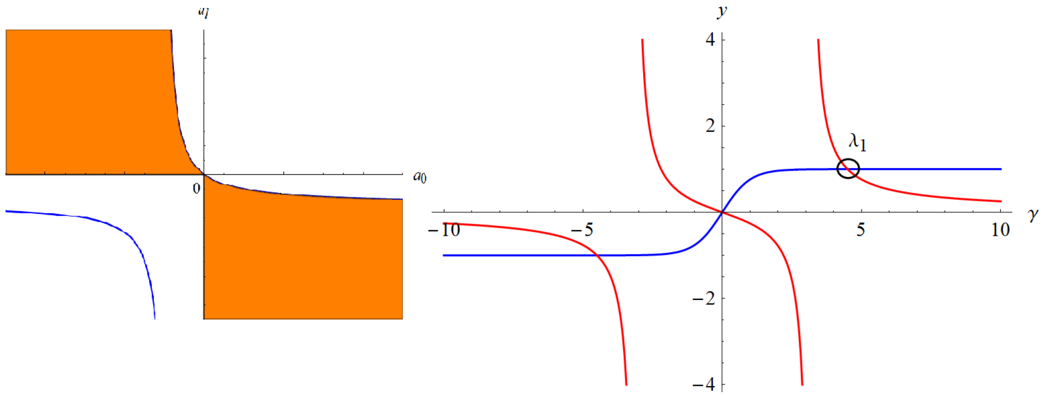

Figure 4: Choosing any two values for a_0 and a_l in the shaded region of the a_0a_l-plane gives us the two plots on the right. The shaded region is the union of $a_0 < 0$ and $a_l > 0$ and $a_0 + a_l < -a_0a_l l$ as well as $a_0 > 0$ and $a_l < 0$ and $a_0 + a_l < -a_0a_l l$, which means physically (usually) that there is more absorption than radiation at the boundaries. In blue is a plot of $y = \tanh \gamma l$ and in red is a plot of $y = -[(a_0 + a_l)\gamma]/(\gamma^2 + a_0a_l)$. There is one intersection and hence one eigenvalue λ_1 in this region.

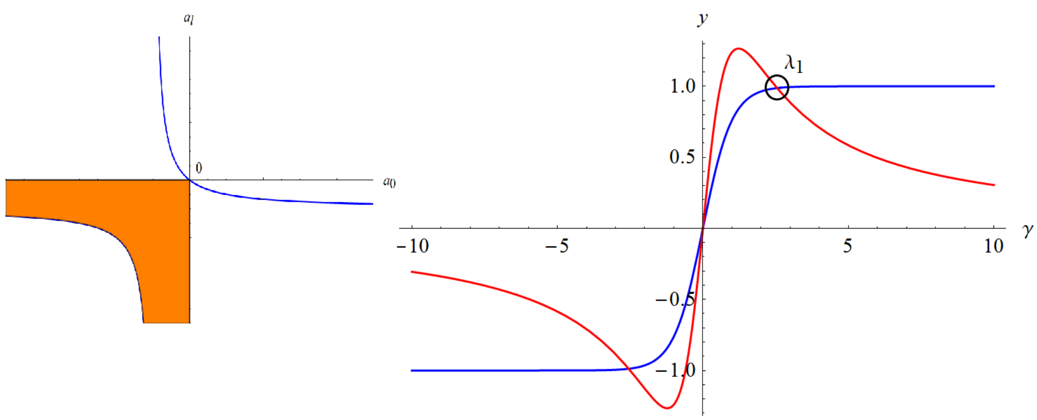

Figure 5: Choosing any two values for a_0 and a_l in the shaded region of the a_0a_l-plane gives us the two plots on the right. The shaded region is $a_0 < 0$ and $a_l < 0$ and $a_0 + a_l < -a_0a_l l$, which means physically that there is absorption at both boundaries. In blue is a plot of $y = \tanh \gamma l$ and in red is a plot of $y = -[(a_0 + a_l)\gamma]/(\gamma^2 + a_0a_l)$. There is one intersection and hence one eigenvalue λ_1 in this region.

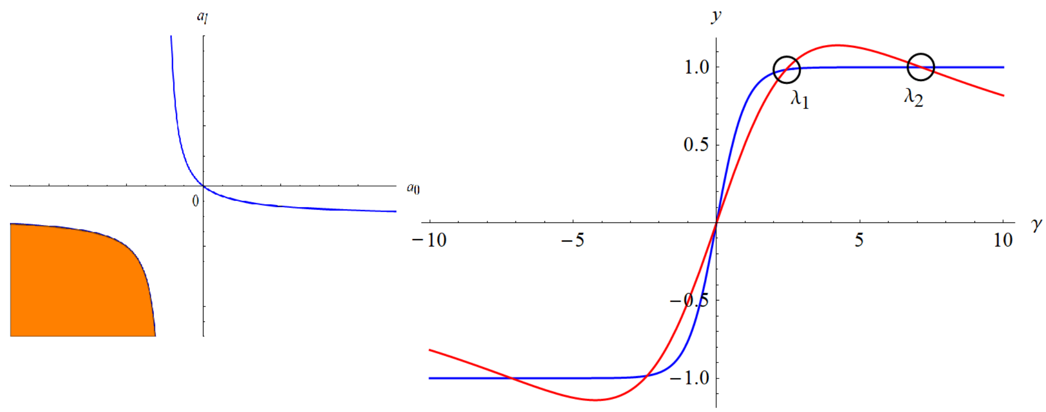

Figure 6: Choosing any two values for a_0 and a_l in the shaded region of the a_0a_l-plane gives us the two plots on the right. The shaded region is $a_0 < 0$ and $a_l < 0$ and $a_0 + a_l > -a_0a_l l$, which means physically that there is absorption at both boundaries. In blue is a plot of $y = \tanh \gamma l$ and in red is a plot of $y = -[(a_0 + a_l)\gamma]/(\gamma^2 + a_0a_l)$. There are two intersections and hence two eigenvalues, λ_1 and λ_2, in this region.

Therefore, we have the following picture of the a_0a_l-plane.

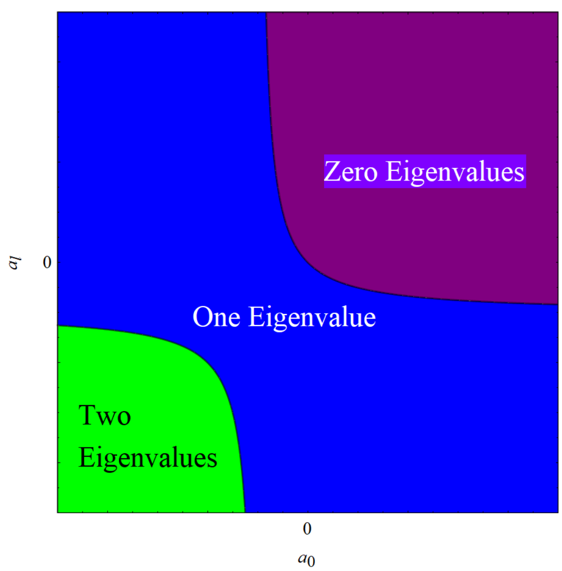

Figure 7: The hyperbola $a_0 + a_l = -a_0a_l l$ separates the regions in the a_0a_l-plane where there are zero, one, and two negative eigenvalues.

Part (c)

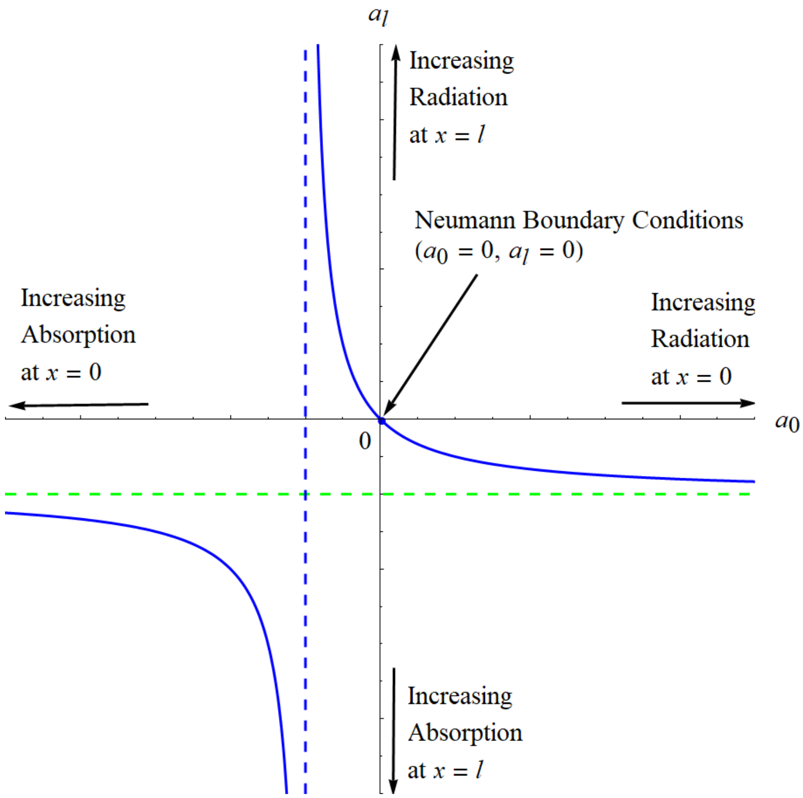

Figure 8: To get $X'(0) = 0$ and $X'(l) = 0$ in the Robin boundary conditions, we set $a_0 = 0$ and $a_l = 0$. Also, positive a_0 and a_l represent radiation, whereas negative a_0 and a_l represent absorption.

Part (d)

Solve the Robin boundary conditions for $X(0)$ and $X(l)$.

$$X'(0) - a_0 X(0) = 0 \quad \rightarrow \quad X(0) = \frac{X'(0)}{a_0}$$

$$X'(l) + a_l X(l) = 0 \quad \rightarrow \quad X(l) = -\frac{X'(l)}{a_l}$$

In order to get $X(0) = 0$ and $X(l) = 0$, we require $a_0 \to \pm\infty$ and $a_l \to \pm\infty$. Therefore, the Dirichlet boundary conditions are at the four corners of the $a_0 a_l$-plane.