Exercise 15

Find the equation for the eigenvalues $\lambda$ of the problem

$$(\kappa(x)X')' + \lambda \rho(x)X = 0 \quad \text{for } 0 < x < l \text{ with } X(0) = X(l) = 0,$$

where $\kappa(x) = \kappa_1^2$ for $x < a$, $\kappa(x) = \kappa_2^2$ for $x > a$, $\rho(x) = \rho_1^2$ for $x < a$, and $\rho(x) = \rho_2^2$ for $x > a$. All these constants are positive and $0 < a < l$.

Solution

With $\kappa(x)$ and $\rho(x)$ defined as they are, the ODE above is equivalent to the following system.

$$
\begin{cases}
(\kappa_1^2 X')' + \lambda \rho_1^2 X = 0 & x < a \\
(\kappa_2^2 X')' + \lambda \rho_2^2 X = 0 & x > a \\
\kappa_1^2 X'' + \lambda \rho_1^2 X = 0 & x < a \\
\kappa_2^2 X'' + \lambda \rho_2^2 X = 0 & x > a \\
X'' = -\lambda \frac{\rho_1^2}{\kappa_1^2} X & x < a \\
X'' = -\lambda \frac{\rho_2^2}{\kappa_2^2} X & x > a
\end{cases}
$$

The solution to the ODE depends on whether $\lambda$ is positive, zero, or negative.

Determination of Positive Eigenvalues: $\lambda = \mu^2$

Assuming $\lambda$ is positive, the system of differential equations for $X$ becomes

$$
\begin{cases}
X'' = -\mu^2 \frac{\rho_1^2}{\kappa_1^2} X & x < a \\
X'' = -\mu^2 \frac{\rho_2^2}{\kappa_2^2} X & x > a
\end{cases}
$$

The solution to the system can be written in terms of sine and cosine.

$$X(x) = \begin{cases} 
C_1 \cos \mu \frac{\rho_1}{\kappa_1} x + C_2 \sin \mu \frac{\rho_1}{\kappa_1} x & x < a \\
C_3 \cos \mu \frac{\rho_2}{\kappa_2} x + C_4 \sin \mu \frac{\rho_2}{\kappa_2} x & x > a
\end{cases}$$

The boundary condition, $X(0) = 0$, applies to the first case, and the boundary condition, $X(l) = 0$, applies to the second case.

$$X(0) = C_1 = 0$$

$$X(l) = C_3 \cos \mu \frac{\rho_2}{\kappa_2} l + C_4 \sin \mu \frac{\rho_2}{\kappa_2} l \quad \rightarrow \quad C_3 = -C_4 \frac{\sin \mu \frac{\rho_2}{\kappa_2} l}{\cos \mu \frac{\rho_2}{\kappa_2} l}$$
So the solution becomes

\[
X(x) = \begin{cases}
C_2 \sin \mu_{\kappa_1} \frac{\rho_1}{\kappa_1} x & x < a \\
-C_4 \sin \mu_{\kappa_2} \frac{\rho_1}{\kappa_2} l + C_4 \sin \mu_{\kappa_2} \frac{\rho_2}{\kappa_2} x & x > a
\end{cases}
\]

\[
= \begin{cases}
C_2 \sin \mu_{\kappa_1} \frac{\rho_1}{\kappa_1} x & x < a \\
C_4 \left( \sin \mu_{\kappa_2} \frac{\rho_2}{\kappa_2} x - \frac{\sin \mu_{\kappa_2} \frac{\rho_1}{\kappa_2} l}{\cos \mu_{\kappa_2} \frac{\rho_2}{\kappa_2}} \cos \mu_{\kappa_2} \frac{\rho_2}{\kappa_2} x \right) & x > a
\end{cases}
\]

Replace \( C_4 \) with a new integration constant, \( C_5 \cos \mu_{\kappa_2} \frac{\rho_2}{\kappa_2} l \), to simplify the second expression.

\[
= \begin{cases}
C_2 \sin \mu_{\kappa_1} \frac{\rho_1}{\kappa_1} x & x < a \\
C_5 \left( \sin \mu_{\kappa_2} \frac{\rho_2}{\kappa_2} x \cos \mu_{\kappa_2} \frac{\rho_2}{\kappa_2} l - \sin \mu_{\kappa_2} \frac{\rho_2}{\kappa_2} \cos \mu_{\kappa_2} \frac{\rho_2}{\kappa_2} x \right) & x > a
\end{cases}
\]

The expression in parentheses is the angle subtraction formula for sine.

\[
= \begin{cases}
C_2 \sin \mu_{\kappa_1} \frac{\rho_1}{\kappa_1} x & x < a \\
C_5 \sin \mu_{\kappa_2} \frac{\rho_2}{\kappa_2} (x - l) & x > a
\end{cases}
\]

To figure out the two remaining constants, we require the eigenfunction and its slope to be continuous at \( x = a \). That is,

\[
X(a-) = X(a+) \rightarrow C_2 \sin \mu \frac{\rho_1}{\kappa_1} a = C_5 \sin \mu \frac{\rho_2}{\kappa_2} (a - l)
\]

\[
X'(a-) = X'(a+) \rightarrow C_2 k_{\kappa_1} \frac{\rho_1}{\kappa_1} \cos \mu \frac{\rho_1}{\kappa_1} a = C_5 k_{\kappa_2} \frac{\rho_2}{\kappa_2} \cos \mu \frac{\rho_2}{\kappa_2} (a - l)
\]

Divide the first equation by the second one to obtain the equation for \( \mu \).

\[
\frac{\kappa_1}{\rho_1} \tan \mu \frac{\rho_1}{\kappa_1} a = \frac{\kappa_2}{\rho_2} \tan \mu \frac{\rho_2}{\kappa_2} (a - l)
\]

This is a transcendental equation, so we leave it as it is. To get the answer at the back of the book, factor out a minus sign from the argument of tangent,

\[
\frac{\kappa_1}{\rho_1} \tan \mu \frac{\rho_1}{\kappa_1} a = \frac{\kappa_2}{\rho_2} \tan \mu \frac{\rho_2}{\kappa_2} (l - a),
\]

invert both sides,

\[
\frac{\rho_1}{\kappa_1} \cot \mu \frac{\rho_1}{\kappa_1} a = \frac{\rho_2}{\kappa_2} \cot \mu \frac{\rho_2}{\kappa_2} (l - a),
\]

and then bring all terms to the left side.

\[
\frac{\rho_1}{\kappa_1} \cot \mu \frac{\rho_1}{\kappa_1} a + \frac{\rho_2}{\kappa_2} \cot \mu \frac{\rho_2}{\kappa_2} (l - a) = 0
\]

(1)

www.stemjock.com
Figure 1: This is a plot of the function on the left side of equation (1) with $\rho_1/\kappa_1 = 1/2$, $a = 3$, $\rho_2/\kappa_2 = 1/4$, and $l = 5$. The values of $\mu$ that solve the equation are the zeroes of the function. Since the eigenvalues are given by $\lambda = \mu^2$ and cotangent is an odd function, negative values of $\mu$ give redundant values for $\lambda$.

**Determination of the Zero Eigenvalue: $\lambda = 0$**

Assuming $\lambda$ is zero, the system of differential equations for $X$ becomes

\[
\begin{cases}
X'' = 0 & x < a \\
X'' = 0 & x > a
\end{cases}
\]

The solution to the system is a linear function.

\[X(x) = \begin{cases} 
C_6 x + C_7 & x < a \\
C_8 x + C_9 & x > a
\end{cases}\]

The boundary condition, $X(0) = 0$, applies to the first case, and the boundary condition, $X(l) = 0$, applies to the second case.

\[
\begin{align*}
X(0) &= C_7 = 0 \\
X(l) &= C_8 l + C_9 = 0 \quad \rightarrow \quad C_9 = -C_8 l
\end{align*}
\]

So the solution becomes

\[X(x) = \begin{cases} 
C_6 x & x < a \\
C_8 x - C_8 l & x > a
\end{cases}\]

= \begin{cases} 
C_6 x & x < a \\
C_8 (x - l) & x > a
\end{cases} .

To figure out the two remaining constants, we require the eigenfunction and its slope to be continuous at $x = a$. That is,

\[
\begin{align*}
X(a-) &= X(a+) \quad \rightarrow \quad C_6 a = C_8 (a - l) \\
X'(a-) &= X'(a+) \quad \rightarrow \quad C_6 = C_8.
\end{align*}
\]

www.stemjock.com
These two equations imply that $C_6 = 0$ and $C_8 = 0$, which means only the trivial solution is obtained from considering $\lambda = 0$. Hence, zero is not an eigenvalue.

**Determination of Negative Eigenvalues: $\lambda = -\eta^2$**

Assuming $\lambda$ is negative, the system of differential equations for $X$ becomes

$$
\begin{cases}
X'' = \eta^2 \frac{\rho_1^2}{\kappa_1^2} X & x < a \\
X'' = \eta^2 \frac{\rho_2^2}{\kappa_2^2} X & x > a
\end{cases}
$$

The solution to the system can be written in terms of hyperbolic sine and hyperbolic cosine.

$$
X(x) = \begin{cases}
C_{10} \cosh \eta \frac{\rho_1}{\kappa_1} x + C_{11} \sinh \eta \frac{\rho_1}{\kappa_1} x & x < a \\
C_{12} \cosh \eta \frac{\rho_2}{\kappa_2} x + C_{13} \sinh \eta \frac{\rho_2}{\kappa_2} x & x > a
\end{cases}
$$

The boundary condition, $X(0) = 0$, applies to the first case, and the boundary condition, $X(l) = 0$, applies to the second case.

$$
X(0) = C_{10} = 0
$$

$$
X(l) = C_{12} \cosh \eta \frac{\rho_2}{\kappa_2} l + C_{13} \sinh \eta \frac{\rho_2}{\kappa_2} l \quad \rightarrow \quad C_{12} = -C_{13}\frac{\sinh \eta \frac{\rho_2}{\kappa_2} l}{\cosh \eta \frac{\rho_2}{\kappa_2} l}
$$

So the solution becomes

$$
X(x) = \begin{cases}
C_{11} \sinh \eta \frac{\rho_1}{\kappa_1} x & x < a \\
-C_{13}\frac{\sinh \eta \frac{\rho_2}{\kappa_2} l}{\cosh \eta \frac{\rho_2}{\kappa_2} l} \cosh \eta \frac{\rho_2}{\kappa_2} x + C_{13} \sinh \eta \frac{\rho_2}{\kappa_2} x & x > a
\end{cases}
$$

$$
= \begin{cases}
C_{11} \sinh \eta \frac{\rho_1}{\kappa_1} x & x < a \\
C_{13} \left( \sinh \eta \frac{\rho_2}{\kappa_2} x - \frac{\sinh \eta \frac{\rho_2}{\kappa_2} l}{\cosh \eta \frac{\rho_2}{\kappa_2} l} \cosh \eta \frac{\rho_2}{\kappa_2} x \right) & x > a
\end{cases}
$$

Replace $C_{13}$ with a new integration constant, $C_{14} \cosh \eta \frac{\rho_2}{\kappa_2} l$, to simplify the second expression.

$$
= \begin{cases}
C_{11} \sinh \eta \frac{\rho_1}{\kappa_1} x & x < a \\
C_{14} \left( \sinh \eta \frac{\rho_2}{\kappa_2} x \cosh \eta \frac{\rho_2}{\kappa_2} l - \sinh \eta \frac{\rho_2}{\kappa_2} l \cosh \eta \frac{\rho_2}{\kappa_2} x \right) & x > a
\end{cases}
$$

The expression in parentheses is the angle subtraction formula for hyperbolic sine.

$$
= \begin{cases}
C_{11} \sinh \eta \frac{\rho_1}{\kappa_1} x & x < a \\
C_{14} \sinh \eta \frac{\rho_2}{\kappa_2} (x - l) & x > a
\end{cases}
$$

To figure out the two remaining constants, we require the eigenfunction and its slope to be continuous at $x = a$. That is,

$$
X(a-) = X(a+) \quad \rightarrow \quad C_{11} \sinh \eta \frac{\rho_1}{\kappa_1} a = C_{14} \sinh \eta \frac{\rho_2}{\kappa_2} (a - l)
$$

$$
X'(a-) = X'(a+) \quad \rightarrow \quad C_{11} \eta \frac{\rho_1}{\kappa_1} \cosh \eta \frac{\rho_1}{\kappa_1} a = C_{14} \eta \frac{\rho_2}{\kappa_2} \cosh \eta \frac{\rho_2}{\kappa_2} (a - l).
$$
Divide the first equation by the second one to obtain the equation for $\eta$.

$$\frac{\kappa_1}{\rho_1} \tanh \eta \frac{\rho_1}{\kappa_1} a = \frac{\kappa_2}{\rho_2} \tanh \eta \frac{\rho_2}{\kappa_2} (a - l)$$

Factor a minus sign from the argument of hyperbolic tangent,

$$\frac{\kappa_1}{\rho_1} \tanh \eta \frac{\rho_1}{\kappa_1} a = -\frac{\kappa_2}{\rho_2} \tanh \eta \frac{\rho_2}{\kappa_2} (l - a),$$

invert both sides,

$$\frac{\rho_1}{\kappa_1} \coth \eta \frac{\rho_1}{\kappa_1} a = -\frac{\rho_2}{\kappa_2} \coth \eta \frac{\rho_2}{\kappa_2} (l - a),$$

and bring both terms to the left side.

$$\frac{\rho_1}{\kappa_1} \coth \eta \frac{\rho_1}{\kappa_1} a + \frac{\rho_2}{\kappa_2} \coth \eta \frac{\rho_2}{\kappa_2} (l - a) = 0 \quad (2)$$

Figure 2: This is a plot of the function on the left side of equation (2) with $\rho_1/\kappa_1 = 1/2$, $a = 3$, $\rho_2/\kappa_2 = 1/4$, and $l = 5$. The values of $\eta$ that solve the equation are the zeroes of the function. Since there are none, there are no negative eigenvalues.

Therefore, the eigenvalues are $\lambda = \mu^2$, where $\mu$ is obtained from the equation,

$$\frac{\rho_1}{\kappa_1} \cot \mu \frac{\rho_1}{\kappa_1} a + \frac{\rho_2}{\kappa_2} \cot \mu \frac{\rho_2}{\kappa_2} (l - a) = 0.$$