Exercise 2
Consider the eigenvalue problem with Robin BCs at both ends:

\[-X'' = \lambda X\]
\[X'(0) - a_0 X(0) = 0, \quad X'(l) + a_l X(l) = 0.\]

(a) Show that \(\lambda = 0\) is an eigenvalue if and only if \(a_0 + a_l = -a_0 a_l l\).

(b) Find the eigenfunctions corresponding to the zero eigenvalue. (Hint: First solve the ODE for \(X(x)\). The solutions are not sines or cosines.)

Solution

Part (a)
If \(\lambda = 0\), then the ODE simplifies to

\[-X'' = 0.\]

The general solution for \(X\) is a linear function.

\[X(x) = C_1 x + C_2\]

Apply the two provided boundary conditions to determine \(C_1\) and \(C_2\).

\[X'(0) - a_0 X(0) = C_1 - a_0 C_2 = 0 \quad (1)\]
\[X'(l) + a_l X(l) = C_1 + a_l (C_1 l + C_2) = 0 \quad (2)\]

Solving equation (1) for \(C_1\) gives \(C_1 = a_0 C_2\). Substitute this into equation (2).

\[a_0 C_2 + a_l (a_0 C_2 l + C_2) = 0\]

Divide both sides by \(C_2\).

\[a_0 + a_l (a_0 l + 1) = 0\]

Expand the left side.

\[a_0 + a_l a_0 l + a_l = 0\]

Therefore, zero is an eigenvalue if and only if \(a_0 + a_l = -a_0 a_l l\) is satisfied.

Part (b)
The eigenfunctions associated with the zero eigenvalue are

\[X(x) = C_1 x + C_2\]
\[= (a_0 C_2) x + C_2\]
\[= C_2 (a_0 x + 1).\]

Therefore,

\[X_0(x) = a_0 x + 1.\]