Exercise 3

Derive the eigenvalue equation (16) for the negative eigenvalues $\lambda = -\gamma^2$ and the formula (17) for the eigenfunctions.

Solution

The ODE under consideration is

$$-X'' = \lambda X$$

subject to the boundary conditions

$$X'(0) - a_0 X(0) = 0$$
$$X'(l) + a_l X(l) = 0$$

The aim is to find the negative eigenvalues $\lambda = -\gamma^2$ and the corresponding eigenfunctions.

$$X'' = \gamma^2 X$$

This is a linear homogeneous ODE with constant coefficients, so the solution is of the form

$$X(x) = e^{rx}.$$

This will be substituted into the ODE to determine r. Before doing so, though, find the derivatives of $X(x)$.

$$X = e^{rx} \quad \rightarrow \quad X' = re^{rx} \quad \rightarrow \quad X'' = r^2 e^{rx}$$

Now we're ready to make the substitutions.

$$r^2 e^{rx} = \gamma^2 e^{rx}$$

Divide both sides by e^{rx}.

$$r^2 = \gamma^2$$

Solve this equation for r by factoring.

$$r^2 - \gamma^2 = 0 \quad \rightarrow \quad (r + \gamma)(r - \gamma) = 0$$

Thus,

$$r = \{\pm\gamma\},$$

and the solution to the ODE is

$$X(x) = C_1 e^{-\gamma x} + C_2 e^{\gamma x}.$$

With the help of Euler's formula,

$$e^{i\gamma x} = \cos \gamma x + i \sin \gamma x,$$

the solution can be written in terms of sine and cosine.

$$X(x) = C_1(\cos i\gamma x + i \sin i\gamma x) + C_2(\cos i\gamma x - i \sin i\gamma x)$$

Make use of the identities, $\cosh \gamma x = \cos i\gamma x$ and $\sinh \gamma x = -i \sin i\gamma x$.

$$\begin{aligned} X(x) &= C_1(\cosh \gamma x - \sinh \gamma x) + C_2(\cosh \gamma x + \sinh \gamma x) \\ &= (C_1 + C_2) \cosh \gamma x + (-C_1 + C_2) \sinh \gamma x \end{aligned}$$

Use new constants of integration here to obtain the general solution of the ODE.

$$X(x) = C_3 \cosh \gamma x + C_4 \sinh \gamma x$$

Apply the boundary conditions now to determine C_3 and C_4.

$$\begin{aligned} X'(0) - a_0 X(0) &= \gamma C_4 - a_0 C_3 = 0 \\ X'(l) + a_l X(l) &= \gamma(C_3 \sinh \gamma l + C_4 \cosh \gamma l) + a_l(C_3 \cosh \gamma l + C_4 \sinh \gamma l) = 0 \end{aligned}$$

Solve the first equation for C_4

$$C_4 = \frac{a_0}{\gamma} C_3$$

and plug it into the second equation.

$$\gamma \left(C_3 \sinh \gamma l + \frac{a_0}{\gamma} C_3 \cosh \gamma l \right) + a_l \left(C_3 \cosh \gamma l + \frac{a_0}{\gamma} C_3 \sinh \gamma l \right) = 0$$

Expand the left side and cancel out C_3.

$$\gamma \sinh \gamma l + a_0 \cosh \gamma l + a_l \cosh \gamma l + \frac{a_0 a_l}{\gamma} \sinh \gamma l = 0$$

Factor $\sinh \gamma l$ and $\cosh \gamma l$.

$$\left(\gamma + \frac{a_0 a_l}{\gamma} \right) \sinh \gamma l + (a_0 + a_l) \cosh \gamma l = 0$$

$$\left(\frac{\gamma^2 + a_0 a_l}{\gamma} \right) \sinh \gamma l = -(a_0 + a_l) \cosh \gamma l$$

Therefore, the negative eigenvalues are $\lambda = -\gamma^2$, where γ is a solution to

$$\tanh \gamma l = -\frac{(a_0 + a_l)\gamma}{\gamma^2 + a_0 a_l}.$$

The eigenfunctions associated with these eigenvalues are

$$\begin{aligned} X(x) &= C_3 \cosh \gamma x + C_4 \sinh \gamma x \\ &= C_3 \cosh \gamma x + \frac{a_0}{\gamma} C_3 \sinh \gamma x \\ &= C_3 \left(\cosh \gamma x + \frac{a_0}{\gamma} \sinh \gamma x \right). \end{aligned}$$

Therefore,

$$X(x) = \cosh \gamma x + \frac{a_0}{\gamma} \sinh \gamma x.$$