Exercise 6

If $a_0 = a_l = a$ in the Robin problem, show that:

(a) There are *no* negative eigenvalues if $a \geq 0$, there is *one* if $-2/l < a < 0$, and there are *two* if $a < -2/l$.

(b) Zero is an eigenvalue if and only if $a = 0$ or $a = -2/l$.

Solution

The eigenvalue problem to solve here is

$$-X'' = \lambda X$$

subject to the Robin boundary conditions,

$$\begin{aligned} X'(0) - aX(0) &= 0 \\ X'(l) + aX(l) &= 0. \end{aligned}$$

Part (a)

Suppose the eigenvalues are negative $\lambda = -\gamma^2$. Then

$$X'' = \gamma^2 X.$$

The solution to this differential equation can be written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \gamma x + C_2 \sinh \gamma x$$

Apply the two boundary conditions here to determine C_1 and C_2.

$$\begin{aligned} X'(0) - aX(0) &= \gamma C_2 - aC_1 = 0 \\ X'(l) + aX(l) &= \gamma(C_1 \sinh \gamma l + C_2 \cosh \gamma l) + a(C_1 \cosh \gamma l + C_2 \sinh \gamma l) = 0 \end{aligned}$$

The first equation can be solved for C_2.

$$C_2 = \frac{a}{\gamma} C_1$$

Substitute this result into the second equation.

$$\gamma \left(C_1 \sinh \gamma l + \frac{a}{\gamma} C_1 \cosh \gamma l \right) + a \left(C_1 \cosh \gamma l + \frac{a}{\gamma} C_1 \sinh \gamma l \right) = 0$$

Divide both sides by $\gamma C_1 \cosh \gamma l$.

$$\tanh \gamma l + \frac{a}{\gamma} + \frac{a}{\gamma} + \frac{a^2}{\gamma^2} \tanh \gamma l = 0$$

$$\left(1 + \frac{a^2}{\gamma^2} \right) \tanh \gamma l = -\frac{2a}{\gamma}$$

$$(\gamma^2 + a^2) \tanh \gamma l = -2a\gamma$$

Divide both sides by $\gamma^2 + a^2$.

$$\tanh \gamma l = -\frac{2a\gamma}{\gamma^2 + a^2}$$

Let $\mu = \gamma l$ in the equation to remove l from the argument of tanh. Then $\gamma = \mu/l$.

$$\tanh \mu = -\frac{2a\frac{\mu}{l}}{\frac{\mu^2}{l^2} + a^2}$$

Multiply the numerator and denominator of the right side by l^2 and group a and l together. Thus, the negative eigenvalues are

$$\lambda = -\frac{\mu^2}{l^2},$$

where μ satisfies

$$\tanh \mu = -\frac{2(al)\mu}{\mu^2 + (al)^2}.$$

The solutions to this transcendental equation occur where the graphs of these two functions intersect. If $a = 0$, then the equation simplifies to $\tanh \mu = 0$. To solve for μ, write $\tanh \mu$ in terms of the exponential function.

$$\tanh \mu = \frac{e^{\mu} - e^{-\mu}}{e^{\mu} + e^{-\mu}} = 0 \quad \rightarrow \quad e^{\mu} - e^{-\mu} = 0 \quad \rightarrow \quad e^{\mu} = e^{-\mu} \quad \rightarrow \quad \mu = 0$$

So there are no negative eigenvalues when $a = 0$. Suppose now that $a > 0$. Since $\tanh \mu$ and $-2(al)\mu/[\mu^2 + (al)^2]$ are odd functions of μ and λ is a function of μ^2, negative values of μ that satisfy the equation yield redundant values of λ. The point is that only intersections for $\mu > 0$ need to be considered. The hyperbolic tangent function is positive and the rational function is negative for $\mu > 0$, so there are no intersections. Therefore, there are no eigenvalues for $a \geq 0$.

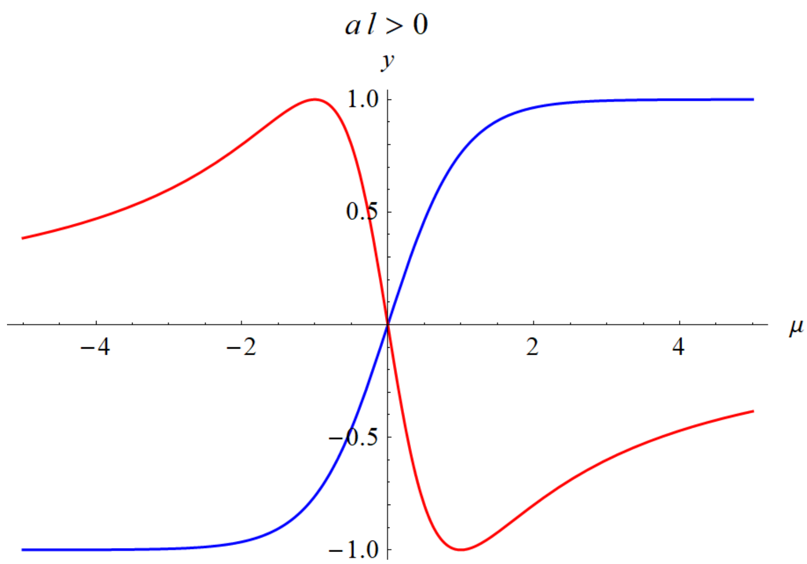

Figure 1: In blue is a plot of $y = \tanh \mu$, and in red is a plot of $y = -2(al)\mu/[\mu^2 + (al)^2]$. There are no intersections for $\mu > 0$, which means there are no negative eigenvalues.

Now we will show that there's one intersection (and hence one eigenvalue) if $-2 < al < 0$ and two intersections (and hence two eigenvalues) if $al < -2$. The derivative of $\tanh\mu$ is $(\operatorname{sech}\mu)^2$, so $\tanh\mu$ is an increasing function for all μ. Writing it in terms of exponentials,

$$\tanh\mu = \frac{e^{\mu} - e^{-\mu}}{e^{\mu} + e^{-\mu}} = \frac{1 - e^{-2\mu}}{1 + e^{-2\mu}},$$

we see that if $\mu = 0$, then $\tanh\mu = 0$. In addition, as $\mu \to \infty$, $\tanh\mu \to 1$. Shifting our attention now to the rational function, the first derivative of it is (after simplifying)

$$\frac{d}{d\mu}\left[-\frac{2(al)\mu}{\mu^2 + (al)^2}\right] = \frac{2(al)[\mu^2 - (al)^2]}{[\mu^2 + (al)^2]^2},$$

which means the rational function is

$$\begin{cases} \text{increasing} & \text{if } \mu < -(al) \\ \text{decreasing} & \text{if } \mu > -(al) \end{cases}.$$

The extrema occur when the first derivative is zero, that is, when $\mu = \pm(al)$.

$$\left.-\frac{2(al)\mu}{\mu^2 + (al)^2}\right|_{\mu=-(al)} = 1 \quad \text{(maximum)}$$
$$\left.-\frac{2(al)\mu}{\mu^2 + (al)^2}\right|_{\mu=(al)} = -1 \quad \text{(minimum)}$$

The rational function goes to 1 at a finite value of μ unlike the hyperbolic tangent function. If the hyperbolic tangent function increases faster than the rational function initially, then there will be two intersections. If it does not, then there will only be one intersection.

$$\left.\frac{d}{d\mu}(\tanh\mu)\right|_{\mu=0} = 1$$
$$\left.\frac{d}{d\mu}\left[-\frac{2(al)\mu}{\mu^2 + (al)^2}\right]\right|_{\mu=0} = \frac{2}{-al}$$

We see that if $-al > 2$, or $al < -2$, then there are two intersections. If $-al < 2$, or $-2 < al < 0$, then there is only one intersection. These ideas are illustrated in Figure 2 and Figure 3.

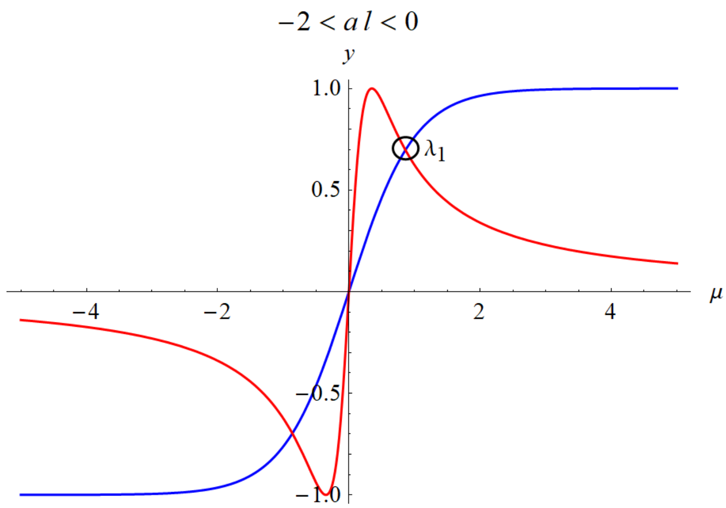

Figure 2: In blue is a plot of $y = \tanh\mu$ and in red is a plot of $y = -2(al)\mu/[\mu^2 + (al)^2]$. If $-2 < al < 0$, then the hyperbolic tangent function does not increase faster than the rational function initially, and there is only one eigenvalue, λ_1.

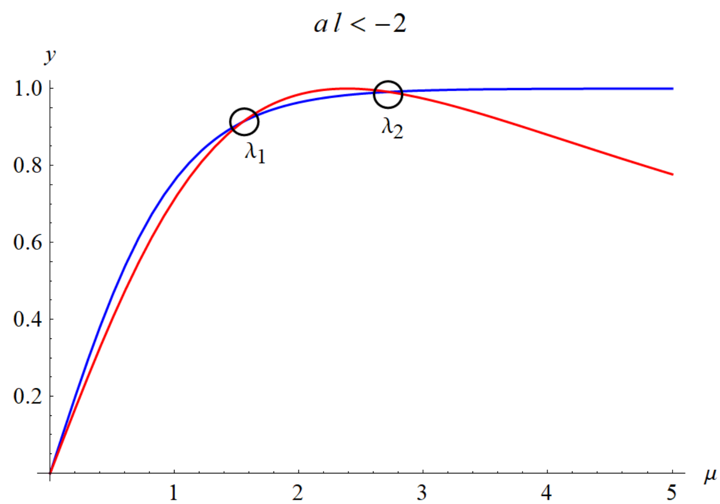

Figure 3: In blue is a plot of $y = \tanh\mu$ and in red is a plot of $y = -2(al)\mu/[\mu^2 + (al)^2]$. If $al < -2$, then the hyperbolic tangent function increases faster than the rational function initially, and there are two eigenvalues, λ_1 and λ_2.

Part (b)

Suppose that $\lambda = 0$. Then the differential equation for X simplifies to

$$X'' = 0,$$

which has the general solution

$$X(x) = C_3 x + C_4.$$

Apply the boundary conditions here to determine C_3 and C_4.

$$\begin{aligned} X'(0) - aX(0) &= C_3 - aC_4 = 0 \\ X'(l) + aX(l) &= C_3 + a(C_3 l + C_4) = 0 \end{aligned}$$

Solve the first equation for C_3

$$C_3 = aC_4$$

and plug it into the second equation.

$$aC_4 + a(aC_4 l + C_4) = 0$$

Divide both sides by C_4 and expand the left side.

$$a^2 l + 2a = 0$$

Factor a.

$$a(al + 2) = 0$$

Therefore, zero is an eigenvalue if and only if $a = 0$ or $al + 2 = 0$, that is, $a = -2/l$.