Exercise 7

If $a_0 = a_l = a$, show that as $a \to +\infty$, the eigenvalues tend to the eigenvalues of the Dirichlet problem. That is,

$$\lim_{a\to\infty}\left\{\beta_n(a) - \frac{(n+1)\pi}{l}\right\} = 0,$$

where $\lambda_n(a) = [\beta_n(a)]^2$ is the $(n+1)$st eigenvalue.

Solution

The eigenvalue problem under consideration here is

$$-X'' = \lambda X$$

subject to the Robin boundary conditions,

$$\begin{aligned} X'(0) - a_0 X(0) &= 0 \\ X'(l) + a_l X(l) &= 0. \end{aligned}$$

If $a_0 = a_l = a$, then the boundary conditions become

$$\begin{aligned} X'(0) - aX(0) &= 0 \\ X'(l) + aX(l) &= 0. \end{aligned}$$

Solve the first and second equations for $X(0)$ and $X(l)$, respectively.

$$\begin{aligned} X(0) &= \frac{X'(0)}{a} \\ X(l) &= -\frac{X'(l)}{a} \end{aligned}$$

Now take the limit as $a \to \infty$.

$$\begin{aligned} X(0) &= 0 \\ X(l) &= 0 \end{aligned}$$

These are Dirichlet boundary conditions. Therefore, as $a \to \infty$ the eigenvalues tend to those of the corresponding Dirichlet problem. We will calculate them now.

Determination of Positive Eigenvalues: $\lambda = \beta^2$

Assuming the eigenvalues are positive, $\lambda = \beta^2$, the differential equation becomes

$$-X'' = \beta^2 X.$$

Its solution can be written in terms of sine and cosine.

$$X(x) = C_1 \cos\beta x + C_2 \sin\beta x$$

Apply the boundary conditions here to determine C_1 and C_2.

$$\begin{aligned} X(0) &= C_1 = 0 \\ X(l) &= C_2 \sin\beta l = 0 \end{aligned}$$

To obtain a nontrivial solution for $X(x)$, we insist that $C_2 \neq 0$. Then we obtain an equation for the eigenvalues.

$$\sin \beta l = 0 \quad \rightarrow \quad \beta l = n\pi \quad \rightarrow \quad \beta = \frac{n\pi}{l}, \quad n = 1, 2, \ldots$$

So in the limit as $a \rightarrow \infty$, the positive eigenvalues are

$$\lambda = \frac{n^2 \pi^2}{l^2}, \quad n = 1, 2, \ldots.$$

Determination of the Zero Eigenvalue: $\lambda = 0$

If $\lambda = 0$, the differential equation becomes

$$X'' = 0.$$

The general solution can be obtained by integrating both sides with respect to x twice.

$$X(x) = C_3 x + C_4$$

Apply the boundary conditions now to determine C_3 and C_4.

$$X(0) = C_4 = 0$$
$$X(l) = C_3 l = 0$$

These two equations tell us that $C_3 = 0$ and $C_4 = 0$, so only the trivial solution for $X(x)$ is obtained. Thus, in the limit as $a \rightarrow \infty$, zero is not an eigenvalue.

Determination of Negative Eigenvalues: $\lambda = -\gamma^2$

Assuming the eigenvalues are negative, $\lambda = \gamma^2$, the differential equation becomes

$$X'' = \gamma^2 X.$$

Its solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_5 \cosh \gamma x + C_6 \sinh \gamma x$$

Apply the boundary conditions here to determine C_5 and C_6.

$$X(0) = C_5 = 0$$
$$X(l) = C_6 \sinh \gamma l = 0$$

These two equations tell us that $C_5 = 0$ and $C_6 = 0$, so only the trivial solution for $X(x)$ is obtained. Thus, in the limit as $a \rightarrow \infty$, there are no negative eigenvalues.