Exercise 7

If \( a_0 = a_t = a \), show that as \( a \to +\infty \), the eigenvalues tend to the eigenvalues of the Dirichlet problem. That is,

\[
\lim_{a \to \infty} \left\{ \beta_n(a) - \frac{(n+1)\pi}{l} \right\} = 0,
\]

where \( \lambda_n(a) = [\beta_n(a)]^2 \) is the \((n+1)\)st eigenvalue.

Solution

The eigenvalue problem under consideration here is

\[-X'' = \lambda X\]

subject to the Robin boundary conditions,

\[
\begin{align*}
X'(0) - a_0 X(0) &= 0 \\
X'(l) + a_l X(l) &= 0.
\end{align*}
\]

If \( a_0 = a_t = a \), then the boundary conditions become

\[
\begin{align*}
X'(0) - a X(0) &= 0 \\
X'(l) + a X(l) &= 0.
\end{align*}
\]

Solve the first and second equations for \( X(0) \) and \( X(l) \), respectively.

\[
\begin{align*}
X(0) &= \frac{X'(0)}{a} \\
X(l) &= -\frac{X'(l)}{a}
\end{align*}
\]

Now take the limit as \( a \to \infty \).

\[
\begin{align*}
X(0) &= 0 \\
X(l) &= 0
\end{align*}
\]

These are Dirichlet boundary conditions. Therefore, as \( a \to \infty \) the eigenvalues tend to those of the corresponding Dirichlet problem. We will calculate them now.

**Determination of Positive Eigenvalues: \( \lambda = \beta^2 \)**

Assuming the eigenvalues are positive, \( \lambda = \beta^2 \), the differential equation becomes

\[-X'' = \beta^2 X.\]

Its solution can be written in terms of sine and cosine.

\[X(x) = C_1 \cos \beta x + C_2 \sin \beta x\]

Apply the boundary conditions here to determine \( C_1 \) and \( C_2 \).

\[
\begin{align*}
X(0) &= C_1 = 0 \\
X(l) &= C_2 \sin \beta l = 0
\end{align*}
\]
To obtain a nontrivial solution for $X(x)$, we insist that $C_2 \neq 0$. Then we obtain an equation for the eigenvalues.

$$\sin \beta l = 0 \quad \rightarrow \quad \beta l = n\pi \quad \rightarrow \quad \beta = \frac{n\pi}{l}, \quad n = 1, 2, \ldots$$

So in the limit as $a \to \infty$, the positive eigenvalues are

$$\lambda = \frac{n^2\pi^2}{l^2}, \quad n = 1, 2, \ldots.$$ 

**Determination of the Zero Eigenvalue: $\lambda = 0$**

If $\lambda = 0$, the differential equation becomes

$$X'' = 0.$$ 

The general solution can be obtained by integrating both sides with respect to $x$ twice.

$$X(x) = C_3x + C_4$$

Apply the boundary conditions now to determine $C_3$ and $C_4$.

$$X(0) = C_4 = 0$$
$$X(l) = C_3l = 0$$

These two equations tell us that $C_3 = 0$ and $C_4 = 0$, so only the trivial solution for $X(x)$ is obtained. Thus, in the limit as $a \to \infty$, zero is not an eigenvalue.

**Determination of Negative Eigenvalues: $\lambda = -\gamma^2$**

Assuming the eigenvalues are negative, $\lambda = \gamma^2$, the differential equation becomes

$$X'' = \gamma^2 X.$$ 

Its solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_5 \cosh \gamma x + C_6 \sinh \gamma x$$

Apply the boundary conditions here to determine $C_5$ and $C_6$.

$$X(0) = C_5 = 0$$
$$X(l) = C_6 \sinh \gamma l = 0$$

These two equations tell us that $C_5 = 0$ and $C_6 = 0$, so only the trivial solution for $X(x)$ is obtained. Thus, in the limit as $a \to \infty$, there are no negative eigenvalues.