Exercise 2

Let $\phi(x) \equiv x^2$ for $0 \leq x \leq 1 = l$.

(a) Calculate its Fourier sine series.

(b) Calculate its Fourier cosine series.

Solution

Part (a)

The Fourier sine series of our function is defined as

$$x^2 = \sum_{n=1}^{\infty} B_n \sin n\pi x.$$

To solve for B_n, multiply both sides by $\sin m\pi x$ and then integrate both sides from 0 to 1.

$$\begin{aligned}
x^2 &= \sum_{n=1}^{\infty} B_n \sin n\pi x \\
x^2 \sin m\pi x &= \sum_{n=1}^{\infty} B_n \sin n\pi x \sin m\pi x \\
\int_0^1 x^2 \sin m\pi x\, dx &= \int_0^1 \sum_{n=1}^{\infty} B_n \sin n\pi x \sin m\pi x\, dx \\
\int_0^1 x^2 \sin m\pi x\, dx &= \sum_{n=1}^{\infty} B_n \underbrace{\int_0^1 \sin n\pi x \sin m\pi x\, dx}_{\frac{1}{2}\delta_{nm}} \\
\int_0^1 x^2 \sin n\pi x\, dx &= \frac{1}{2} B_n \\
B_n &= 2\frac{-2 + (-1)^n(2 - n^2\pi^2)}{n^3\pi^3}
\end{aligned}$$

Therefore, the Fourier sine series for $\phi(x) = x^2$ over $0 < x < 1$ is

$$x^2 = \sum_{n=1}^{\infty} 2\frac{-2 + (-1)^n(2 - n^2\pi^2)}{n^3\pi^3} \sin n\pi x.$$

Below is a graph of $y = x^2$ and its Fourier sine series (first 30 terms).

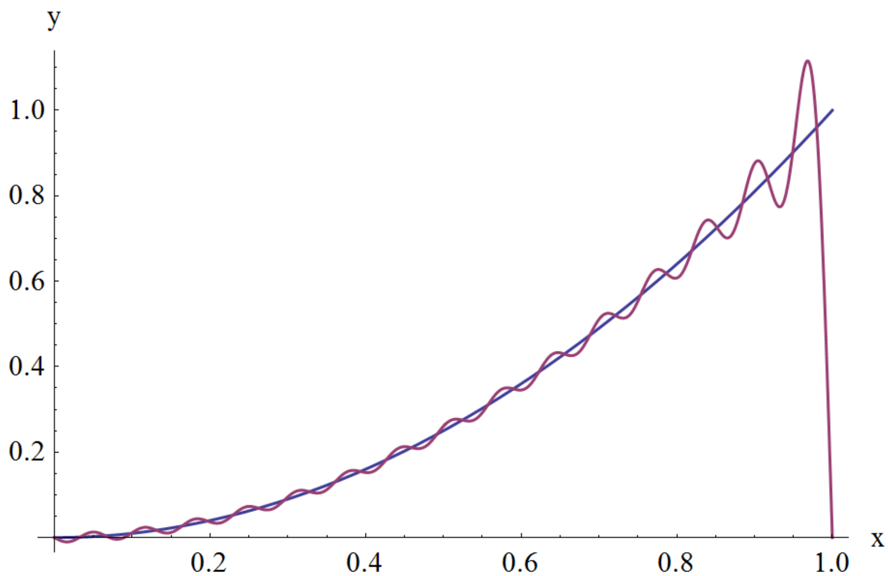

Figure 1: Plot of the partial sum $n = 30$ and $y = x^2$ for $0 < x < 1$.

Part (b)

The Fourier cosine series of our function is defined as

$$x^2 = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos n\pi x.$$

To solve for A_0, simply integrate both sides from 0 to 1.

$$\int_0^1 x^2\,dx = \int_0^1 \frac{1}{2}A_0\,dx + \int_0^1 \sum_{n=1}^{\infty} A_n \cos n\pi x\,dx$$

$$\frac{1}{3} = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \underbrace{\int_0^1 \cos n\pi x\,dx}_{=0}$$

$$A_0 = \frac{2}{3}$$

To solve for A_n, multiply both sides by $\cos m\pi x$ and then integrate both sides from 0 to 1.

$$x^2 \cos m\pi x = \frac{1}{2}A_0 \cos m\pi x + \sum_{n=1}^{\infty} A_n \cos n\pi x \cos m\pi x$$

$$\int_0^1 x^2 \cos m\pi x \, dx = \int_0^1 \frac{1}{2}A_0 \cos m\pi x \, dx + \int_0^1 \sum_{n=1}^{\infty} A_n \cos n\pi x \cos m\pi x \, dx$$

$$\int_0^1 x^2 \cos m\pi x \, dx = \frac{1}{2}A_0 \underbrace{\int_0^1 \cos m\pi x \, dx}_{=0} + \sum_{n=1}^{\infty} A_n \underbrace{\int_0^1 \cos n\pi x \cos m\pi x \, dx}_{=\frac{1}{2}\delta_{nm}}$$

$$\int_0^1 x^2 \cos m\pi x \, dx = \frac{1}{2}A_n$$

$$A_n = \frac{4(-1)^n}{n^2\pi^2}$$

Therefore, the Fourier cosine series for $\phi(x) = x^2$ over $0 < x < 1$ is

$$x^2 = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2\pi^2} \cos n\pi x.$$

Below is a graph of $y = x^2$ and its Fourier cosine series (first 5 terms). Note that the Fourier cosine series converges to x^2 much faster than the Fourier sine series because x^2 and cosine are both even functions.

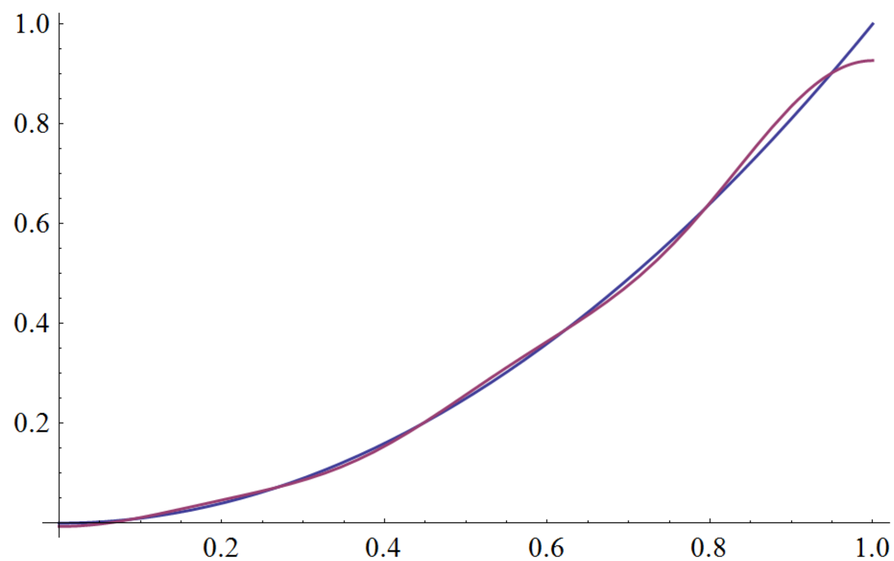

Figure 2: Plot of the partial sum $n = 5$ and $y = x^2$ for $0 < x < 1$.