Exercise 6

(a) By the same method, find the sine series of x^3.

(b) Find the cosine series of x^4.

Solution

Part (a)

From Exercise 5 we have the Fourier sine series expansion of x,

$$\sum_{n=1}^{\infty} \frac{2l}{n\pi}(-1)^{n+1} \sin \frac{n\pi x}{l} = x, \tag{1}$$

and we have the Fourier cosine series expansion of $x^2/2$.

$$\frac{l^2}{6} + \sum_{n=1}^{\infty} \frac{2l^2}{n^2\pi^2}(-1)^n \cos \frac{n\pi x}{l} = \frac{x^2}{2} \tag{2}$$

Integrate both sides of equation (2) with respect to x.

$$\int \left[\frac{l^2}{6} + \sum_{n=1}^{\infty} \frac{2l^2}{n^2\pi^2}(-1)^n \cos \frac{n\pi x}{l}\right] dx = \int \frac{x^2}{2}\, dx$$

Split the integral on the left side into two and evaluate the integral on the right side.

$$\int \frac{l^2}{6}\, dx + \int \sum_{n=1}^{\infty} \frac{2l^2}{n^2\pi^2}(-1)^n \cos \frac{n\pi x}{l}\, dx = \frac{x^3}{6} + C_1$$

Evaluate the first integral on the left and bring the constants in front of the other one.

$$\frac{l^2}{6}x + \sum_{n=1}^{\infty} \frac{2l^2}{n^2\pi^2}(-1)^n \int \cos \frac{n\pi x}{l}\, dx = \frac{x^3}{6} + C_1$$

Evaluate the last integral.

$$\frac{l^2}{6}x + \sum_{n=1}^{\infty} \frac{2l^2}{n^2\pi^2}(-1)^n \cdot \frac{l}{n\pi} \sin \frac{n\pi x}{l} = \frac{x^3}{6} + C_1$$

$$\frac{l^2}{6}x + \sum_{n=1}^{\infty} \frac{2l^3}{n^3\pi^3}(-1)^n \sin \frac{n\pi x}{l} = \frac{x^3}{6} + C_1 \tag{3}$$

This equation holds for all values of x, so the constant C_1 can be determined by setting x to something convenient. Set $x = l$.

$$\frac{l^2}{6}l + \sum_{n=1}^{\infty} \frac{2l^3}{n^3\pi^3}(-1)^n \underbrace{\sin n\pi}_{=\,0} = \frac{l^3}{6} + C_1$$

$$\frac{l^3}{6} = \frac{l^3}{6} + C_1$$

Thus,

$$C_1 = 0.$$

Substitute this result for C_1 into equation (3).

$$\frac{l^2}{6}x + \sum_{n=1}^{\infty} \frac{2l^3}{n^3\pi^3}(-1)^n \sin\frac{n\pi x}{l} = \frac{x^3}{6}$$

Multiply both sides by 6.

$$l^2 x + \sum_{n=1}^{\infty} \frac{12l^3}{n^3\pi^3}(-1)^n \sin\frac{n\pi x}{l} = x^3$$

Substitute the Fourier sine series expansion for x from equation (1).

$$l^2 \sum_{n=1}^{\infty} \frac{2l}{n\pi}(-1)^{n+1} \sin\frac{n\pi x}{l} + \sum_{n=1}^{\infty} \frac{12l^3}{n^3\pi^3}(-1)^n \sin\frac{n\pi x}{l} = x^3$$

$$-\sum_{n=1}^{\infty} \frac{2l^3}{n\pi}(-1)^n \sin\frac{n\pi x}{l} + \sum_{n=1}^{\infty} \frac{12l^3}{n^3\pi^3}(-1)^n \sin\frac{n\pi x}{l} = x^3$$

Combine the series.

$$\sum_{n=1}^{\infty} \left(\frac{12l^3}{n^3\pi^3} - \frac{2l^3}{n\pi}\right)(-1)^n \sin\frac{n\pi x}{l} = x^3$$

$$\sum_{n=1}^{\infty} \frac{2l^3}{n^3\pi^3}\left(6 - n^2\pi^2\right)(-1)^n \sin\frac{n\pi x}{l} = x^3$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{2l^3}{n^3\pi^3}\left(n^2\pi^2 - 6\right)(-1)^{n+1} \sin\frac{n\pi x}{l} = x^3.$$

Part (b)

To find the Fourier cosine series of x^4, integrate both sides of the previous equation with respect to x.

$$\int \sum_{n=1}^{\infty} \frac{2l^3}{n^3\pi^3}\left(n^2\pi^2 - 6\right)(-1)^{n+1} \sin\frac{n\pi x}{l}\,dx = \int x^3\,dx$$

Evaluate the integral on the right side and bring the constants in front of the one on the left.

$$\sum_{n=1}^{\infty} \frac{2l^3}{n^3\pi^3}\left(n^2\pi^2 - 6\right)(-1)^{n+1} \int \sin\frac{n\pi x}{l}\,dx = \frac{x^4}{4} + C_2$$

Evaluate the last integral.

$$\sum_{n=1}^{\infty} \frac{2l^3}{n^3\pi^3}\left(n^2\pi^2 - 6\right)(-1)^{n+1} \cdot \frac{l}{n\pi}\left(-\cos\frac{n\pi x}{l}\right) = \frac{x^4}{4} + C_2$$

$$\sum_{n=1}^{\infty} \frac{2l^4}{n^4\pi^4}\left(n^2\pi^2 - 6\right)(-1)^n \cos\frac{n\pi x}{l} = \frac{x^4}{4} + C_2 \tag{4}$$

This equation holds for all values of x, so the constant C_2 can be determined by setting x to something convenient. Set $x = l$.

$$\sum_{n=1}^{\infty} \frac{2l^4}{n^4\pi^4}\left(n^2\pi^2 - 6\right)(-1)^n \underbrace{\cos n\pi}_{=\,(-1)^n} = \frac{l^4}{4} + C_2$$

$$\sum_{n=1}^{\infty} \frac{2l^4}{n^4\pi^4}\left(n^2\pi^2 - 6\right) = \frac{l^4}{4} + C_2$$

Expand the summand.

$$\sum_{n=1}^{\infty} \left(\frac{2l^4}{n^2\pi^2} - \frac{12l^4}{n^4\pi^4}\right) = \frac{l^4}{4} + C_2$$

Split up the infinite series into two and bring the constants in front of them.

$$\frac{2l^4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} - \frac{12l^4}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{l^4}{4} + C_2$$

Each of these infinite series is known: the one with $1/n^2$ is equal to $\pi^2/6$ and the one with $1/n^4$ is equal to $\pi^4/90$.

$$\frac{2l^4}{\pi^2}\cdot\frac{\pi^2}{6} - \frac{12l^4}{\pi^4}\cdot\frac{\pi^4}{90} = \frac{l^4}{4} + C_2$$

$$\frac{l^4}{3} - \frac{2l^4}{15} = \frac{l^4}{4} + C_2$$

Thus,

$$\begin{aligned} C_2 &= l^4\left(\frac{1}{3} - \frac{2}{15} - \frac{1}{4}\right) \\ &= -\frac{l^4}{20}. \end{aligned}$$

Plug this result for C_2 into equation (4).

$$\sum_{n=1}^{\infty} \frac{2l^4}{n^4\pi^4}\left(n^2\pi^2 - 6\right)(-1)^n \cos\frac{n\pi x}{l} = \frac{x^4}{4} - \frac{l^4}{20}$$

Bring the constant to the left side.

$$\frac{l^4}{20} + \sum_{n=1}^{\infty} \frac{2l^4}{n^4\pi^4}\left(n^2\pi^2 - 6\right)(-1)^n \cos\frac{n\pi x}{l} = \frac{x^4}{4}$$

Therefore,

$$\frac{l^4}{5} + \sum_{n=1}^{\infty} \frac{8l^4}{n^4\pi^4}\left(n^2\pi^2 - 6\right)(-1)^n \cos\frac{n\pi x}{l} = x^4.$$