Exercise 9

Solve $u_{tt} = c^2 u_{xx}$ for $0 < x < \pi$, with the boundary conditions $u_x(0,t) = u_x(\pi,t) = 0$ and the initial conditions $u(x,0) = 0$, $u_t(x,0) = \cos^2 x$. (*Hint:* See (4.2.7).)

Solution

The PDE and its boundary conditions are linear and homogeneous, so the method of separation of variables can be applied. Assume a product solution of the form $u(x,t) = X(x)T(t)$ and plug it into the PDE

$$u_{tt} = c^2 u_{xx} \quad \to \quad XT'' = c^2 X'' T$$

and the boundary conditions.

$$\begin{aligned} u_x(0,t) &= 0 & &\to & X'(0)T(t) &= 0 & &\to & X'(0) &= 0 \\ u_x(\pi,t) &= 0 & &\to & X'(\pi)T(t) &= 0 & &\to & X'(\pi) &= 0 \end{aligned}$$

Now separate variables in the PDE: bring all functions of t and constants to the left side and all functions of x to the right side. The final answer would be the same if all constants were brought to the right side.

$$\frac{T''}{c^2 T} = \frac{X''}{X}$$

The only way a function of t on the left can be equal to a function of x on the right is if both sides are equal to a constant λ.

$$\frac{T''}{c^2 T} = \frac{X''}{X} = \lambda$$

Values of λ for which $X'(0) = 0$ and $X'(\pi) = 0$ are satisfied are called the eigenvalues, and the nontrivial functions $X(x)$ associated with them are called the eigenfunctions.

Determination of Positive Eigenvalues: $\lambda = \mu^2$

Assuming λ is positive, the differential equation for X becomes

$$\frac{X''}{X} = \mu^2.$$

Multiply both sides by X.

$$X'' = \mu^2 X$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \mu x + C_2 \sinh \mu x$$

Now use the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} X'(0) &= \mu C_2 = 0 \\ X'(\pi) &= \mu(C_1 \sinh \mu\pi + C_2 \cosh \mu\pi) = 0 \end{aligned}$$

The first equation tells us that $C_2 = 0$. Consequently, the only way the second equation can be satisfied is if $C_1 = 0$. Only the trivial solution $X(x) = 0$ results from considering positive values for λ, so there are no positive eigenvalues.

Determination of the Zero Eigenvalue: $\lambda = 0$

Assuming λ is zero, the differential equation for X becomes

$$\frac{X''}{X} = 0.$$

Multiply both sides by X.

$$X'' = 0$$

The general solution is a linear function.

$$X(x) = C_3 x + C_4$$

Now use the boundary conditions to determine C_3 and C_4.

$$X'(0) = C_3 = 0$$
$$X'(\pi) = C_3 = 0$$

We see that C_4 remains arbitrary. The eigenfunction then is $X(x) = C_4$, so zero is in fact an eigenvalue. The ODE for $T(t)$ will now be solved.

$$\frac{T''}{c^2 T} = 0$$

Multiply both sides by $c^2 T$.

$$T'' = 0$$

The general solution to this equation is also a linear function.

$$T(t) = C_5 t + C_6$$

Determination of Negative Eigenvalues: $\lambda = -\gamma^2$

Assuming λ is negative, the differential equation for X becomes

$$\frac{X''}{X} = -\gamma^2.$$

Multiply both sides by X.

$$X'' = -\gamma^2 X$$

The general solution can be written in terms of sine and cosine.

$$X(x) = C_7 \cos \gamma x + C_8 \sin \gamma x$$

Apply the boundary conditions here to determine C_7 and C_8.

$$X'(0) = \gamma C_8 = 0$$
$$X'(\pi) = \gamma(-C_7 \sin \gamma\pi + C_8 \cos \gamma\pi) = 0$$

The first equation gives $C_8 = 0$, so the second equation reduces to

$$-\gamma C_7 \sin \gamma\pi = 0.$$

In order to avoid getting the trivial solution, we insist that $C_7 \neq 0$. The equation for γ is then

$$-\gamma \sin \gamma\pi = 0$$

$$\sin \gamma\pi = 0$$

$$\gamma\pi = n\pi \quad \rightarrow \quad \gamma = n.$$

The eigenfunctions associated with these eigenvalues are

$$X(x) = C_7 \cos \gamma x \quad \rightarrow \quad X_n(x) = \cos nx.$$

The ODE for $T(t)$ will now be solved.

$$\frac{T''}{c^2 T} = -\gamma^2$$

Multiply both sides by $c^2 T$.

$$T'' = -c^2\gamma^2 T$$

The general solution can be written in terms of sine and cosine.

$$T(t) = C_9 \cos c\gamma t + C_{10} \sin c\gamma t \quad \rightarrow \quad T_n(t) = C_9 \cos cnt + C_{10} \sin cnt$$

According to the principle of linear superposition, the solution to the PDE for $u(x,t)$ is a linear combination of all products $T_n(t)X_n(x)$ over all the eigenvalues.

$$u(x,t) = A + Bt + \sum_{n=1}^{\infty} (D_n \cos cnt + E_n \sin cnt) \cos nx$$

Now we will use the provided initial conditions to determine the coefficients, A, B, D_n, and E_n.

$$u(x,0) = A + \sum_{n=1}^{\infty} D_n \cos nx = 0$$

From this equation, we can say that $A = 0$ and $D_n = 0$. Take the derivative with respect to t of the general solution to use the second initial condition.

$$u_t(x,t) = B + \sum_{n=1}^{\infty} (-cnD_n \sin cnt + cnE_n \cos cnt) \cos nx$$

Plug in $t = 0$ and use the second initial condition.

$$u_t(x,0) = B + \sum_{n=1}^{\infty} (cnE_n) \cos nx = \cos^2 x \tag{1}$$

To determine B, integrate both sides of equation (1) with respect to x over the domain the PDE is defined.

$$\int_0^{\pi} \left[B + \sum_{n=1}^{\infty} (cnE_n) \cos nx \right] dx = \int_0^{\pi} \cos^2 x \, dx$$

Split up the integral on the left side into two and use the trigonometric identity, $\cos^2 x = \frac{1}{2}(1+\cos 2x)$, on the right side.

$$\int_0^\pi B\,dx + \int_0^\pi \sum_{n=1}^\infty (cnE_n)\cos nx\,dx = \int_0^\pi \frac{1}{2}(1+\cos 2x)\,dx$$

Evaluate the first integral and bring the constants in front of the other integrals.

$$B\cdot\pi + \sum_{n=1}^\infty (cnE_n)\int_0^\pi \cos nx\,dx = \frac{1}{2}\int_0^\pi (1+\cos 2x)\,dx$$

Evaluate the remaining integrals.

$$B\cdot\pi + \sum_{n=1}^\infty (cnE_n)\cdot \underbrace{\frac{1}{n}\sin nx\bigg|_0^\pi}_{=\,0} = \frac{1}{2}\bigg(\pi + \underbrace{\frac{1}{2}\sin 2x\bigg|_0^\pi}_{=\,0}\bigg)\,dx$$

$$B\cdot\pi = \frac{1}{2}(\pi)$$

Consequently,

$$B = \frac{1}{2}.$$

To determine E_n, multiply both sides of equation (1) by $\cos mx$, where m is a positive integer.

$$B\cos mx + \sum_{n=1}^\infty (cnE_n)\cos nx\cos mx = \cos^2 x\cos mx$$

Integrate both sides with respect to x over the domain the PDE is defined.

$$\int_0^\pi \left[B\cos mx + \sum_{n=1}^\infty (cnE_n)\cos nx\cos mx\right]dx = \int_0^\pi \cos^2 x\cos mx\,dx$$

Split up the integral on the left side into two and use the trigonometric identity, $\cos^2 x = \frac{1}{2}(1+\cos 2x)$, on the right side.

$$\int_0^\pi B\cos mx\,dx + \int_0^\pi \sum_{n=1}^\infty (cnE_n)\cos nx\cos mx\,dx = \int_0^\pi \frac{1}{2}(1+\cos 2x)\cos mx\,dx$$

Bring the constants in front of the integrals.

$$B\int_0^\pi \cos mx\,dx + \sum_{n=1}^\infty (cnE_n)\int_0^\pi \cos nx\cos mx\,dx = \frac{1}{2}\int_0^\pi (1+\cos 2x)\cos mx\,dx$$

If $n \neq m$, then the second integral on the left is equal to 0 thanks to the orthogonality of the trigonometric functions. This can be verified with the product-to-sum formula for cosine. When $n = m$, the integrand becomes $\cos^2 nx$, and the result of the integral is $\pi/2$.

$$B\int_0^\pi \cos nx\,dx + (cnE_n)\cdot\frac{\pi}{2} = \frac{1}{2}\int_0^\pi (1+\cos 2x)\cos nx\,dx$$

Evaluate the integral on the left side and split up the integral on the right side. The first one is the same as the integral on the left side.

$$\underbrace{\frac{B}{n}\sin nx\bigg|_0^\pi}_{=0} + (cnE_n)\cdot\frac{\pi}{2} = \frac{1}{2}\left(\underbrace{\int_0^\pi \cos nx\,dx}_{=0} + \int_0^\pi \cos 2x\cos nx\,dx\right)$$

Multiply both sides by 2 and divide both sides by $cn\pi$ to solve for E_n.

$$E_n = \frac{1}{cn\pi}\int_0^\pi \cos 2x\cos nx\,dx$$

As noted before, the integral here is 0 for $n \neq 2$. Only when $n = 2$ is the integral nonzero ($\pi/2$).

$$E_n = \begin{cases} 0 & n \neq 2 \\ \dfrac{1}{4c} & n = 2 \end{cases}$$

The general solution for $u(x,t)$ simplifies to

$$u(x,t) = \frac{t}{2} + E_2 \sin 2ct\cos 2x.$$

Therefore,

$$u(x,t) = \frac{t}{2} + \frac{1}{4c}\sin 2ct\cos 2x.$$