Exercise 2

Let \( \phi(x) \equiv x^2 \) for \( 0 \leq x \leq 1 = l \).

(a) Calculate its Fourier sine series.

(b) Calculate its Fourier cosine series.

Solution

Part (a)

The Fourier sine series of our function is defined as

\[
x^2 = \sum_{n=1}^{\infty} B_n \sin n\pi x.
\]

To solve for \( B_n \), multiply both sides by \( \sin m\pi x \) and then integrate both sides from 0 to 1.

\[
\int_0^1 x^2 \sin m\pi x \, dx = \int_0^1 \sum_{n=1}^{\infty} B_n \sin n\pi x \sin m\pi x \, dx
\]

\[
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\]

\[
\int_0^1 \sin n\pi x \sin m\pi x \, dx = \frac{\delta_{nm}}{2n^3}\pi^3
\]

\[
B_n = 2\frac{-2 + (-1)^n(2 - n^2\pi^2)}{n^3\pi^3}
\]

Therefore, the Fourier sine series for \( \phi(x) = x^2 \) over \( 0 < x < 1 \) is

\[
x^2 = \sum_{n=1}^{\infty} 2\frac{-2 + (-1)^n(2 - n^2\pi^2)}{n^3\pi^3} \sin n\pi x.
\]

Below is a graph of \( y = x^2 \) and its Fourier sine series (first 30 terms).
Figure 1: Plot of the partial sum \( n = 30 \) and \( y = x^2 \) for \( 0 < x < 1 \).

**Part (b)**

The Fourier cosine series of our function is defined as

\[
x^2 = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \cos n\pi x.
\]

To solve for \( A_0 \), simply integrate both sides from 0 to 1.

\[
\int_0^1 x^2 \, dx = \int_0^1 \frac{1}{2} A_0 \, dx + \int_0^1 \sum_{n=1}^{\infty} A_n \cos n\pi x \, dx
\]

\[
\frac{1}{3} = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \int_0^1 \cos n\pi x \, dx
\]

\[
A_0 = \frac{2}{3}
\]
To solve for $A_n$, multiply both sides by $\cos m\pi x$ and then integrate both sides from 0 to 1.

$$x^2 \cos m\pi x = \frac{1}{2} A_0 \cos m\pi x + \sum_{n=1}^{\infty} A_n \cos n\pi x \cos m\pi x$$

$$\int_0^1 x^2 \cos m\pi x \, dx = \int_0^1 \frac{1}{2} A_0 \cos m\pi x \, dx + \int_0^1 \sum_{n=1}^{\infty} A_n \cos n\pi x \cos m\pi x \, dx$$

$$\int_0^1 x^2 \cos m\pi x \, dx = \frac{1}{2} A_0 \underbrace{\int_0^1 \cos m\pi x \, dx}_{=0} + \sum_{n=1}^{\infty} A_n \underbrace{\int_0^1 \cos n\pi x \cos m\pi x \, dx}_{=\frac{1}{2} \delta_{nm}}$$

$$\int_0^1 x^2 \cos m\pi x \, dx = \frac{1}{2} A_n$$

$$A_n = \frac{4(-1)^n}{n^2 \pi^2}$$

Therefore, the Fourier cosine series for $\phi(x) = x^2$ over $0 < x < 1$ is

$$x^2 = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2 \pi^2} \cos n\pi x.$$ 

Below is a graph of $y = x^2$ and its Fourier cosine series (first 5 terms). Note that the Fourier cosine series converges to $x^2$ much faster than the Fourier sine series because $x^2$ and cosine are both even functions.

Figure 2: Plot of the partial sum $n = 5$ and $y = x^2$ for $0 < x < 1$. 

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