Exercise 3

Consider the function $\phi(x) \equiv x$ on $(0, l)$. On the same graph, *sketch* the following functions.

(a) The sum of the first three (nonzero) terms of its Fourier sine series.

(b) The sum of the first three (nonzero) terms of its Fourier cosine series.

Solution

Part (a)

Assume that x has a Fourier sine series expansion with coefficients B_n to be determined.

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} = x$$

To solve the equation for B_n, multiply both sides by $\sin(m\pi x/l)$, where m is a positive integer.

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} = x \sin \frac{m\pi x}{l}$$

Now integrate both sides with respect to x over the domain $\phi(x)$ is defined.

$$\int_0^l \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l}\, dx = \int_0^l x \sin \frac{m\pi x}{l}\, dx$$

Bring the constants in front of the integral.

$$\sum_{n=1}^{\infty} B_n \int_0^l \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l}\, dx = \int_0^l x \sin \frac{m\pi x}{l}\, dx$$

If $n \neq m$, then the integral on the left is equal to 0 thanks to the orthogonality of the trigonometric functions. This can be verified with the product-to-sum formula for sine. When $n = m$, the integrand becomes $\sin^2(n\pi x/l)$, and the result of the integral is $l/2$.

$$B_n \cdot \frac{l}{2} = \int_0^l x \sin \frac{n\pi x}{l}\, dx$$

Multiply both sides by $2/l$ to solve for B_n.

$$B_n = \frac{2}{l} \int_0^l x \sin \frac{n\pi x}{l}\, dx$$

Use integration by parts to solve the remaining integral.

$$\begin{aligned}
B_n &= \frac{2}{l}\left[-\frac{l}{n\pi}x\cos\frac{n\pi x}{l}\bigg|_0^l - \int_0^l\left(-\frac{l}{n\pi}\right)\cos\frac{n\pi x}{l}\,dx\right] \\
&= \frac{2}{l}\Bigg[-\frac{l}{n\pi}(l\cos n\pi - 0) + \underbrace{\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\bigg|_0^l}_{=0}\Bigg] \\
&= \frac{2}{l}\left[-\frac{l^2}{n\pi}\cos n\pi\right] \\
&= -\frac{2l}{n\pi}(-1)^n \\
&= \frac{2l}{n\pi}(-1)^{n+1}
\end{aligned}$$

We thus have the Fourier sine series expansion of x.

$$\sum_{n=1}^{\infty}\frac{2l}{n\pi}(-1)^{n+1}\sin\frac{n\pi x}{l} = x$$

Taking only the first three terms of the series, we get an approximation for x.

$$\frac{2l}{\pi}\sin\frac{\pi x}{l} - \frac{l}{\pi}\sin\frac{2\pi x}{l} + \frac{2l}{3\pi}\sin\frac{3\pi x}{l} \approx x$$

Let this partial sum be represented as $f(x)$:

$$f(x) = \frac{2l}{\pi}\sin\frac{\pi x}{l} - \frac{l}{\pi}\sin\frac{2\pi x}{l} + \frac{2l}{3\pi}\sin\frac{3\pi x}{l}.$$

Part (b)

Assume that x has a Fourier cosine series expansion with coefficients A_n to be determined.

$$\sum_{n=0}^{\infty}A_n\cos\frac{n\pi x}{l} = x$$

This is more commonly written as

$$A_0 + \sum_{n=1}^{\infty}A_n\cos\frac{n\pi x}{l} = x. \tag{1}$$

To determine A_0, integrate both sides with respect to x over the domain $\phi(x)$ is defined.

$$\int_0^l\left(A_0 + \sum_{n=1}^{\infty}A_n\cos\frac{n\pi x}{l}\right)dx = \int_0^l x\,dx$$

Split up the integral on the left into two and bring the constants in front of them. Also, evaluate the integral on the right side.

$$A_0\int_0^l dx + \sum_{n=1}^{\infty}A_n\int_0^l\cos\frac{n\pi x}{l}\,dx = \frac{x^2}{2}\bigg|_0^l$$

Evaluate the integrals on the left side.

$$A_0 \cdot l + \sum_{n=1}^{\infty} A_n \cdot \underbrace{\frac{l}{n\pi} \sin \frac{n\pi x}{l}\bigg|_0^l}_{=\,0} = \frac{l^2}{2}$$

We're left with

$$A_0 \cdot l = \frac{l^2}{2},$$

which means

$$A_0 = \frac{l}{2}.$$

To determine A_n, multiply both sides of equation (1) by $\cos(m\pi x/l)$, where m is a positive integer.

$$A_0 \cos \frac{m\pi x}{l} + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l} \cos \frac{m\pi x}{l} = x \cos \frac{m\pi x}{l}$$

Now integrate both sides with respect to x over the domain $\phi(x)$ is defined.

$$\int_0^l \left(A_0 \cos \frac{m\pi x}{l} + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l} \cos \frac{m\pi x}{l} \right) dx = \int_0^l x \cos \frac{m\pi x}{l}\, dx$$

Split up the integral into two and bring the constants in front.

$$A_0 \int_0^l \cos \frac{m\pi x}{l}\, dx + \sum_{n=1}^{\infty} A_n \int_0^l \cos \frac{n\pi x}{l} \cos \frac{m\pi x}{l}\, dx = \int_0^l x \cos \frac{m\pi x}{l}\, dx$$

If $n \neq m$, then the second integral on the left is equal to 0 thanks to the orthogonality of the trigonometric functions. This can be verified with the product-to-sum formula for cosine. When $n = m$, the integrand becomes $\cos^2(n\pi x/l)$, and the result of the integral is $l/2$.

$$A_0 \int_0^l \cos \frac{n\pi x}{l}\, dx + A_n \cdot \frac{l}{2} = \int_0^l x \cos \frac{n\pi x}{l}\, dx$$

Evaluate the remaining integrals. Use integration by parts for the one on the right side.

$$A_0 \cdot \underbrace{\frac{l}{n\pi} \sin \frac{n\pi x}{l}\bigg|_0^l}_{=\,0} + A_n \cdot \frac{l}{2} = \underbrace{\frac{l}{n\pi} x \sin \frac{n\pi x}{l}\bigg|_0^l}_{=\,0} - \int_0^l \frac{l}{n\pi} \sin \frac{n\pi x}{l}\, dx$$

The first term on each side disappears. Evaluate the last integral on the right side.

$$\begin{aligned} A_n \cdot \frac{l}{2} &= \frac{l^2}{n^2\pi^2} \cos \frac{n\pi x}{l}\bigg|_0^l \\ &= \frac{l^2}{n^2\pi^2}(\cos n\pi - 1) \\ &= \frac{l^2}{n^2\pi^2}[(-1)^n - 1] \end{aligned}$$

So then

$$A_n = \frac{2l}{n^2\pi^2}[(-1)^n - 1].$$

We thus have the Fourier cosine series expansion of x.

$$\frac{l}{2} + \sum_{n=1}^{\infty} \frac{2l}{n^2\pi^2}[(-1)^n - 1]\cos\frac{n\pi x}{l} = x$$

Taking only the first three nonzero terms of the series, we get an approximation for x.

$$\frac{l}{2} - \frac{4l}{\pi^2}\cos\frac{\pi x}{l} - \frac{4l}{9\pi^2}\cos\frac{3\pi x}{l} \approx x$$

Let this partial sum be represented as $g(x)$:

$$g(x) = \frac{l}{2} - \frac{4l}{\pi^2}\cos\frac{\pi x}{l} - \frac{4l}{9\pi^2}\cos\frac{3\pi x}{l}.$$

Now we're ready to graph all the functions.

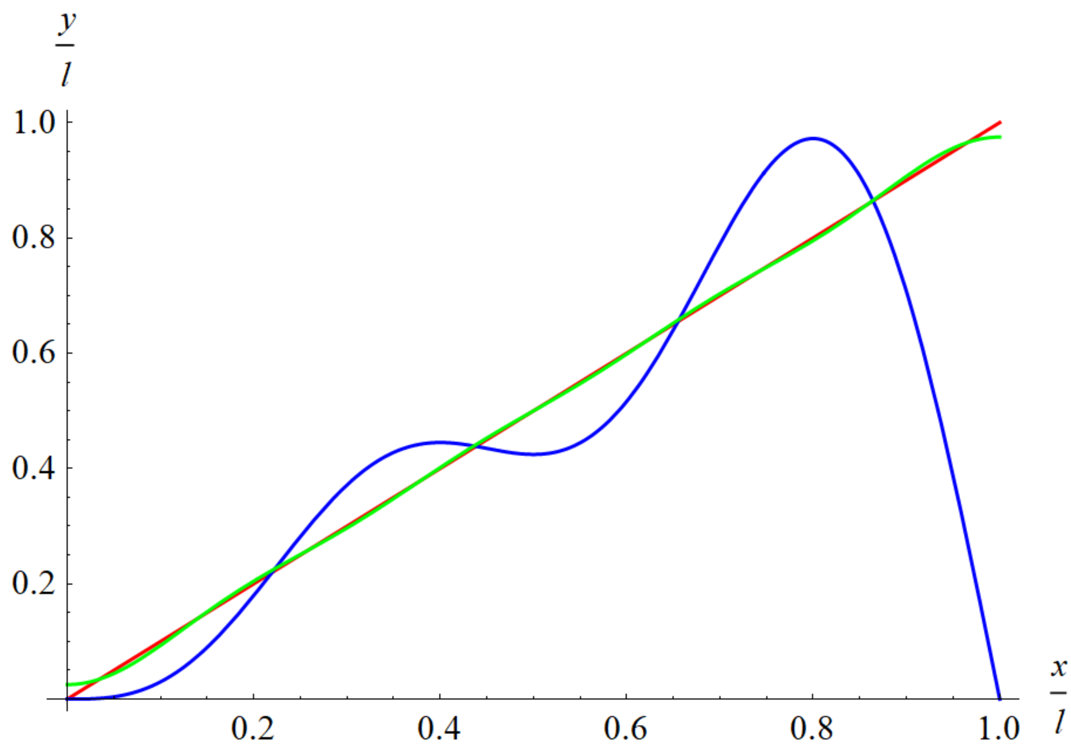

Figure 1: In red is the graph of $y = x$, in blue is the graph of $y = f(x)$, and in green is the graph of $y = g(x)$ for $0 \le x \le l$.

Note that $g(x)$ gives a better approximation to $\phi(x) = x$ than $f(x)$ because the coefficients A_n fall off as $1/n^2$, whereas the coefficients B_n fall off as $1/n$. In other words, the Fourier cosine series converges to $\phi(x) = x$ faster.

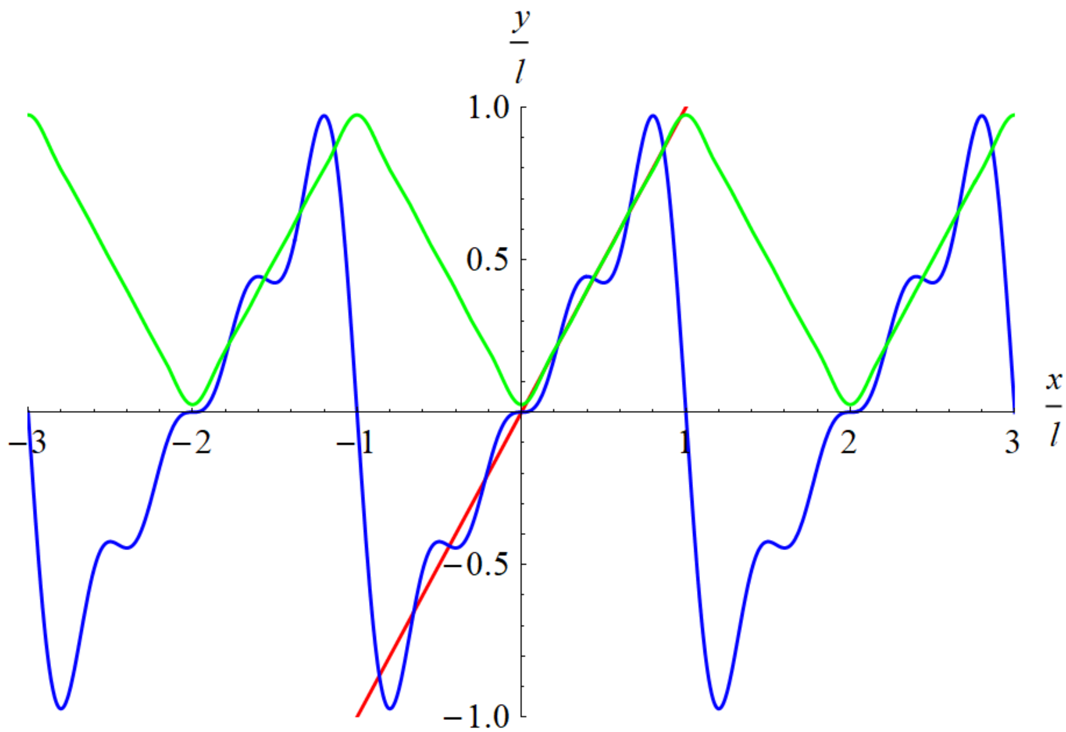

Figure 2: In red is the graph of $y = x$, in blue is the graph of $y = f(x)$, and in green is the graph of $y = g(x)$ for $-3l \leq x \leq 3l$.

It's instructive to see how the graphs look beyond the domain of interest. The Fourier cosine series gives us a periodic function that represents $\phi(x) = x$ for $x \in (0, l)$ and is symmetric about the y-axis (even). On the other hand, the Fourier sine series gives us a periodic function that represents $\phi(x) = x$ for $x \in (0, l)$ and is symmetric about the origin (odd).