Exercise 4

Find the Fourier cosine series of the function $|\sin x|$ in the interval $(-\pi, \pi)$. Use it to find the sums

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}.$$

Solution

Assume that $|\sin x|$ has a Fourier cosine series expansion with coefficients A_n to be determined.

$$\sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{l} = |\sin x|$$

Since $l = \pi$, this can be written as

$$A_0 + \sum_{n=1}^{\infty} A_n \cos nx = |\sin x|. \tag{1}$$

To determine A_0, integrate both sides with respect to x over the domain $|\sin x|$ is defined.

$$\int_{-\pi}^{\pi} \left(A_0 + \sum_{n=1}^{\infty} A_n \cos nx \right) dx = \int_{-\pi}^{\pi} |\sin x|\, dx$$

Split up the integral on the left side into two. In the interval $(-\pi, \pi)$, $\sin x$ is negative from $(-\pi, 0)$ and positive from $(0, \pi)$, so the absolute value sign in the integrand can be removed by splitting the integral on the right side into two over these intervals.

$$\int_{-\pi}^{\pi} A_0\, dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} A_n \cos nx\, dx = \int_{-\pi}^{0} (-\sin x)\, dx + \int_{0}^{\pi} (\sin x)\, dx$$

Evaluate the first integral on the left and those on the right side. Bring the constants in front of the remaining integral.

$$A_0 \cdot 2\pi + \sum_{n=1}^{\infty} A_n \int_{-\pi}^{\pi} \cos nx\, dx = \cos x \Big|_{-\pi}^{0} + (-\cos x)\Big|_{0}^{\pi}$$

Evaluate the last integral.

$$A_0 \cdot 2\pi + \sum_{n=1}^{\infty} A_n \cdot \underbrace{\frac{1}{n} \sin nx \Big|_{-\pi}^{\pi}}_{=\,0} = [1 - (-1)] + [-(-1) + 1]$$

All that remains is

$$A_0 \cdot 2\pi = 4,$$

so

$$A_0 = \frac{2}{\pi}.$$

To determine A_n, multiply both sides of equation (1) by $\cos mx$, where m is a positive integer.

$$A_0 \cos mx + \sum_{n=1}^{\infty} A_n \cos nx \cos mx = |\sin x| \cos mx$$

Now integrate both sides with respect to x over the domain $\phi(x)$ is defined.

$$\int_{-\pi}^{\pi} \left(A_0 \cos mx + \sum_{n=1}^{\infty} A_n \cos nx \cos mx \right) dx = \int_{-\pi}^{\pi} |\sin x| \cos mx \, dx$$

Split up the integral on the left side into two and bring the constants in front.

$$A_0 \int_{-\pi}^{\pi} \cos mx \, dx + \sum_{n=1}^{\infty} A_n \int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \int_{-\pi}^{\pi} |\sin x| \cos mx \, dx$$

If $n \neq m$, then the second integral on the left is equal to 0 thanks to the orthogonality of the trigonometric functions. This can be verified with the product-to-sum formula for cosine. When $n = m$, the integrand becomes $\cos^2 nx$, and the result of the integral is $2\pi/2 = \pi$.

$$A_0 \int_{-\pi}^{\pi} \cos nx \, dx + A_n \cdot \pi = \int_{-\pi}^{\pi} |\sin x| \cos nx \, dx$$

Evaluate the integral on the left side. In the interval $(-\pi, \pi)$, $\sin x$ is negative from $(-\pi, 0)$ and positive from $(0, \pi)$, so the absolute value sign in the integrand can be removed by splitting the integral on the right side into two over these intervals.

$$A_0 \cdot \underbrace{\frac{1}{n} \sin nx \bigg|_{-\pi}^{\pi}}_{=\,0} + A_n \cdot \pi = \int_{-\pi}^{0} (-\sin x) \cos nx \, dx + \int_{0}^{\pi} (\sin x) \cos nx \, dx$$

Make a u-substitution in the first integral.

$$\begin{aligned} u &= -x \quad \rightarrow \quad -u = x \\ du &= -dx \end{aligned}$$

Then we get

$$\begin{aligned} A_n \cdot \pi &= \int_{\pi}^{0} \sin(-u) \cos n(-u) \, du + \int_{0}^{\pi} \sin x \cos nx \, dx \\ &= -\int_{\pi}^{0} \sin u \cos nu \, du + \int_{0}^{\pi} \sin x \cos nx \, dx \\ &= \int_{0}^{\pi} \sin u \cos nu \, du + \int_{0}^{\pi} \sin x \cos nx \, dx \\ &= 2 \int_{0}^{\pi} \sin x \cos nx \, dx. \end{aligned}$$

Divide both sides by π to solve for A_n.

$$A_n = \frac{2}{\pi} \int_{0}^{\pi} \sin x \cos nx \, dx$$

Use a product-to-sum formula to solve the integral.

$$\begin{aligned}
A_n &= \frac{2}{\pi}\int_0^\pi \frac{1}{2}[\sin(x+nx)+\sin(x-nx)]\,dx \\
&= \frac{1}{\pi}\left[\int_0^\pi \sin(1+n)x\,dx + \int_0^\pi \sin(1-n)x\,dx\right] \\
&= \frac{1}{\pi}\left[-\frac{1}{1+n}\cos(1+n)x\bigg|_0^\pi - \frac{1}{1-n}\cos(1-n)x\bigg|_0^\pi\right] \\
&= \frac{1}{\pi}\left\{-\frac{1}{1+n}[(-1)^{1+n}-1] - \frac{1}{1-n}[(-1)^{1-n}-1]\right\}
\end{aligned}$$

$(-1)^{1+n}$ and $(-1)^{1-n}$ yield the same value no matter what n is.

$$\begin{aligned}
&= -\frac{1}{\pi}\left(\frac{1}{1+n}+\frac{1}{1-n}\right)[(-1)^{1+n}-1] \\
&= -\frac{1}{\pi}\left(\frac{2}{1-n^2}\right)[-(-1)^n-1] \\
&= -\frac{2}{\pi}\frac{(-1)^n+1}{n^2-1}
\end{aligned}$$

This formula for A_n only applies if $n \neq 1$. If $n = 1$, then we have for A_n:

$$\begin{aligned}
A_1 &= \frac{2}{\pi}\int_0^\pi \sin x\cos x\,dx \\
&= \frac{2}{\pi}\int_0^\pi \frac{1}{2}\sin 2x\,dx \\
&= \frac{1}{\pi}\int_0^\pi \sin 2x\,dx \\
&= \frac{1}{\pi}\cdot\frac{1}{2}(-\cos 2x)\bigg|_0^\pi \\
&= \frac{1}{2\pi}(-1+1) \\
&= 0.
\end{aligned}$$

The Fourier cosine series for $|\sin x|$ is now known.

$$\frac{2}{\pi} + \sum_{n=2}^\infty \left[-\frac{2}{\pi}\frac{(-1)^n+1}{n^2-1}\right]\cos nx = |\sin x|$$

Notice that if n is odd, then the coefficient of $\cos nx$ is equal to 0. If n is even, then $(-1)^n+1$ is 2. The series can be made to sum over the even integers only by making the substitution $n = 2k$.

$$\frac{2}{\pi} + \sum_{2k=2}^\infty \left[-\frac{2}{\pi}\frac{(-1)^{2k}+1}{(2k)^2-1}\right]\cos 2kx = |\sin x|$$

$$\frac{2}{\pi} + \sum_{k=1}^\infty \left(-\frac{2}{\pi}\frac{2}{4k^2-1}\right)\cos 2kx = |\sin x|$$

Therefore, in the interval $(-\pi, \pi)$

$$\frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos 2kx}{4k^2-1} = |\sin x|. \tag{2}$$

If we set $x = 0$ in equation (2), then we get an equation involving the first series in the problem statement.

$$\frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1} = 0$$

$$-\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1} = -\frac{2}{\pi}$$

Therefore,

$$\sum_{k=1}^{\infty}\frac{1}{4k^2-1} = \frac{1}{2}.$$

If we set $x = \pi/2$ in equation (2), then we get an equation involving the second series in the problem statement.

$$\frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos k\pi}{4k^2-1} = 1$$

$$-\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{4k^2-1} = 1 - \frac{2}{\pi}$$

$$-4\sum_{k=1}^{\infty}\frac{(-1)^k}{4k^2-1} = \pi - 2$$

Therefore,

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{4k^2-1} = -\frac{\pi-2}{4}.$$