Exercise 7

Put $x = 0$ in Exercise 6(b) to deduce the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}.$$

[TYPO: "$n =$" is missing under the summation symbol.]

Solution

From Exercise 6(b) we have the Fourier cosine series expansion of x^4.

$$\frac{l^4}{5} + \sum_{n=1}^{\infty} \frac{8l^4}{n^4\pi^4}\left(n^2\pi^2 - 6\right)(-1)^n \cos\frac{n\pi x}{l} = x^4$$

Set $x = 0$ as instructed.

$$\frac{l^4}{5} + \sum_{n=1}^{\infty} \frac{8l^4}{n^4\pi^4}\left(n^2\pi^2 - 6\right)(-1)^n = 0$$

Divide both sides by l^4 and multiply both sides by 5.

$$1 + \sum_{n=1}^{\infty} \frac{40}{n^4\pi^4}\left(n^2\pi^2 - 6\right)(-1)^n = 0$$

Expand the summand.

$$1 + \sum_{n=1}^{\infty} \left(\frac{40}{n^2\pi^2} - \frac{240}{n^4\pi^4}\right)(-1)^n = 0$$

Split up the infinite series into two and bring the constants in front of them.

$$1 + \frac{40}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} - \frac{240}{\pi^4}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} = 0$$

Bring the first two terms to the right side.

$$-\frac{240}{\pi^4}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} = -\frac{40}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} - 1$$

$$-\frac{240}{\pi^4}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} = \frac{40}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} - 1$$

From part (b) of Exercise 5, we know the value of the series on the right side.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

Plugging this in, we get

$$-\frac{240}{\pi^4}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} = \frac{40}{\pi^2}\cdot\frac{\pi^2}{12} - 1 = \frac{7}{3}.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} = -\frac{7\pi^4}{720}.$$