Exercise 8

A rod has length $l = 1$ and constant $k = 1$. Its temperature satisfies the heat equation. Its left end is held at temperature 0, its right end at temperature 1. Initially (at $t = 0$) the temperature is given by

$$\phi(x) = \begin{cases} \dfrac{5x}{2} & \text{for } 0 < x < \frac{2}{3} \\ 3 - 2x & \text{for } \frac{2}{3} < x < 1. \end{cases}$$

Find the solution, including the coefficients. (*Hint:* First find the equilibrium solution $U(x)$, and then solve the heat equation with initial condition $u(x, 0) = \phi(x) - U(x)$.)

Solution

Let $u = u(x, t)$ represent the temperature. The PDE to solve then is

$$u_t = u_{xx}$$

subject to the boundary conditions,

$$u(0, t) = 0 \quad \text{and} \quad u(1, t) = 1,$$

and the initial condition,

$$u(x, 0) = \phi(x).$$

Because one of the boundary conditions is inhomogeneous, the method of separation of variables cannot be used to solve the PDE as it is. The fact that the boundary conditions are independent of time implies that there will be a state of equilibrium after a long time has passed. If $u_E(x)$ represents the steady state solution, then we expect $u(x, t)$ to converge to $u_E(x)$ in the limit that $t \to \infty$.

$$\lim_{t\to\infty} u(x, t) = u_E(x)$$

We can solve for $u(x, t)$ by taking advantage of the fact that the PDE is linear. Treat it as the sum of an equilibrium part and a transient part: $u(x, t) = u_E(x) + v(x, t)$. At equilibrium u does not change with time, so it satisfies the time-independent PDE and the boundary conditions at $x = 0$ and $x = 1$.

$$\frac{d^2 u_E}{dx^2} = 0, \qquad u_E(0) = 0, \ u_E(1) = 1.$$

Integrate both sides with respect to x.

$$\frac{du_E}{dx} = C_1$$

Integrate both sides with respect to x once more.

$$u_E(x) = C_1 x + C_2$$

Apply the boundary conditions now to determine C_1 and C_2.

$$u_E(0) = C_2 = 0$$
$$u_E(1) = C_1 + C_2 = 1$$

From the last equation we have $C_1 = 1$, so the equilibrium solution is $u_E(x) = x$. The aim now is to determine $v(x,t)$. Write the terms of the PDE, u_t and u_{xx}, in terms of it.

$$\begin{aligned} u_t &= v_t \\ u_x &= u_E' + v_x \\ u_{xx} &= u_E'' + v_{xx} = 0 + v_{xx} = v_{xx} \end{aligned}$$

Hence, the PDE for v is the same as for u.

$$v_t = v_{xx}$$

The initial and boundary conditions for it are obtained as follows.

$$\begin{aligned} u(x,0) = u_E(x) + v(x,0) = x + v(x,0) = \phi(x) \quad &\to \quad v(x,0) = \phi(x) - x \\ u(0,t) = u_E(0) + v(0,t) = 0 + v(0,t) = 0 \quad &\to \quad v(0,t) = 0 \\ u(1,t) = u_E(1) + v(1,t) = 1 + v(1,t) = 1 \quad &\to \quad v(1,t) = 0 \end{aligned}$$

Since the PDE and boundary conditions for v are linear and homogeneous, the method of separation of variables can be applied now. Assume a product solution of the form, $v(x,t) = X(x)T(t)$. The PDE becomes

$$v_t = v_{xx} \quad \to \quad XT' = X''T \quad \to \quad \frac{T'}{T} = \frac{X''}{X}, \tag{1}$$

and the boundary conditions become

$$\begin{aligned} v(0,t) = X(0)T(t) = 0 \quad &\to \quad X(0) = 0 \\ v(1,t) = X(1)T(t) = 0 \quad &\to \quad X(1) = 0. \end{aligned}$$

Regarding equation (1), the only way a function of t on the left can be equal to a function of x on the right is if both sides are equal to a constant λ.

$$\frac{T'}{T} = \frac{X''}{X} = \lambda$$

Values of λ for which $X(0) = 0$ and $X(1) = 0$ are satisfied are called the eigenvalues, and the nontrivial functions $X(x)$ associated with them are called the eigenfunctions.

Determination of Positive Eigenvalues: $\lambda = \mu^2$

Assuming λ is positive, the differential equation for X becomes

$$\frac{X''}{X} = \mu^2.$$

Multiply both sides by X.

$$X'' = \mu^2 X$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_3 \cosh \mu x + C_4 \sinh \mu x$$

Now use the boundary conditions to determine C_3 and C_4.

$$\begin{aligned} X(0) &= C_3 = 0 \\ X(1) &= C_3 \cosh \mu + C_4 \sinh \mu = 0 \end{aligned}$$

The only way the second equation can be satisfied is if $C_4 = 0$. Hence, only the trivial solution $X(x) = 0$ results from considering positive values for λ, and there are no positive eigenvalues.

Determination of the Zero Eigenvalue: $\lambda = 0$

Assuming λ is zero, the differential equation for X becomes

$$\frac{X''}{X} = 0.$$

Multiply both sides by X.

$$X'' = 0$$

The general solution is a linear function.

$$X(x) = C_5 x + C_6$$

Now use the boundary conditions to determine C_5 and C_6.

$$\begin{aligned} X(0) &= C_6 = 0 \\ X(1) &= C_5 + C_6 = 0 \end{aligned}$$

We see that $C_5 = 0$ and $C_6 = 0$. Hence, only the trivial solution $X(x) = 0$ results from considering $\lambda = 0$, and zero is not an eigenvalue.

Determination of Negative Eigenvalues: $\lambda = -\gamma^2$

Assuming λ is negative, the differential equation for X becomes

$$\frac{X''}{X} = -\gamma^2.$$

Multiply both sides by X.

$$X'' = -\gamma^2 X$$

The general solution can be written in terms of sine and cosine.

$$X(x) = C_7 \cos \gamma x + C_8 \sin \gamma x$$

Now use the boundary conditions to determine C_7 and C_8.

$$\begin{aligned} X(0) &= C_7 = 0 \\ X(1) &= C_7 \cos \gamma + C_8 \sin \gamma = 0 \end{aligned}$$

The second equation simplifies to $C_8 \sin \gamma = 0$. To avoid getting the trivial solution, we insist that $C_8 \neq 0$. Doing so yields an equation for the eigenvalues.

$$\sin \gamma = 0$$

$$\gamma = n\pi, \quad n = 1, 2, \ldots$$

The eigenfunctions associated with these eigenvalues are

$$X(x) = C_8 \sin \gamma x \quad \rightarrow \quad X_n(x) = \sin n\pi x.$$

Now solve the differential equation for $T(t)$.

$$\frac{T'}{T} = -\gamma^2$$

The left side is just the derivative of $\ln T$.

$$\frac{d}{dt}(\ln T) = -\gamma^2$$

Integrate both sides with respect to t.

$$\ln T = -\gamma^2 t + C_9$$

Exponentiate both sides.

$$T(t) = e^{-\gamma^2 t + C_9} = e^{-\gamma^2 t} e^{C_9}$$

Use a new constant of integration.

$$T(t) = C_{10} e^{-\gamma^2 t} \quad \rightarrow \quad T_n(t) = e^{-n^2\pi^2 t}$$

According to the principle of linear superposition, the solution to the PDE for $v(x,t)$ is a linear combination of all products $T_n(t)X_n(x)$ over all the eigenvalues.

$$v(x,t) = \sum_{n=1}^{\infty} B_n e^{-n^2\pi^2 t} \sin n\pi x$$

We can use the initial condition now to determine B_n.

$$v(x,0) = \sum_{n=1}^{\infty} B_n \sin n\pi x = \phi(x) - x$$

Multiply both sides by $\sin m\pi x$, where m is a positive integer.

$$\sum_{n=1}^{\infty} B_n \sin n\pi x \sin m\pi x = [\phi(x) - x] \sin m\pi x$$

Integrate both sides with respect to x over the domain the PDE is defined.

$$\int_0^1 \sum_{n=1}^{\infty} B_n \sin n\pi x \sin m\pi x \, dx = \int_0^1 [\phi(x) - x] \sin m\pi x \, dx$$

Bring the constants in front of the integral on the left.

$$\sum_{n=1}^{\infty} B_n \int_0^1 \sin n\pi x \sin m\pi x \, dx = \int_0^1 [\phi(x) - x] \sin m\pi x \, dx$$

If $n \neq m$, then the integral on the left is equal to 0 thanks to the orthogonality of the trigonometric functions. This can be verified with the product-to-sum formula for sine. When $n = m$, the integrand becomes $\sin^2 n\pi x$, and the result of the integral is 1/2.

$$B_n \cdot \frac{1}{2} = \int_0^1 [\phi(x) - x] \sin n\pi x \, dx$$

Multiply both sides by 2 to solve for B_n.

$$B_n = 2 \int_0^1 [\phi(x) - x] \sin n\pi x \, dx$$

Substitute the expression for $\phi(x)$ here. Since it is defined piecewise, the integral will be split into many over each interval.

$$B_n = 2 \left[\int_0^{\frac{2}{3}} \left(\frac{5x}{2} - x \right) \sin n\pi x \, dx + \int_{\frac{2}{3}}^1 (3 - 2x - x) \sin n\pi x \, dx \right]$$

$$B_n = 2 \left[\int_0^{\frac{2}{3}} \frac{3x}{2} \sin n\pi x \, dx + \int_{\frac{2}{3}}^1 (3 - 3x) \sin n\pi x \, dx \right]$$

Split up the second integral into two and bring the constants out in front.

$$\begin{aligned} B_n &= 2 \left(\frac{3}{2} \int_0^{\frac{2}{3}} x \sin n\pi x \, dx + 3 \int_{\frac{2}{3}}^1 \sin n\pi x \, dx - 3 \int_{\frac{2}{3}}^1 x \sin n\pi x \, dx \right) \\ &= 3 \int_0^{\frac{2}{3}} x \sin n\pi x \, dx + 6 \int_{\frac{2}{3}}^1 \sin n\pi x \, dx - 6 \int_{\frac{2}{3}}^1 x \sin n\pi x \, dx \end{aligned}$$

Use integration by parts for the first and third integrals. Evaluate the second one directly.

$$\begin{aligned} &= 3 \left[\frac{x}{n\pi}(-\cos n\pi x) \Big|_0^{\frac{2}{3}} - \int_0^{\frac{2}{3}} \frac{1}{n\pi}(-\cos n\pi x) \, dx \right] + \frac{6}{n\pi}(-\cos n\pi x) \Big|_{\frac{2}{3}}^1 \\ &\qquad - 6 \left[\frac{x}{n\pi}(-\cos n\pi x) \Big|_{\frac{2}{3}}^1 - \int_{\frac{2}{3}}^1 \frac{1}{n\pi}(-\cos n\pi x) \, dx \right] \\ &= 3 \left(-\cancel{\frac{2}{3n\pi} \cos \frac{2n\pi}{3}} + \frac{1}{n^2\pi^2} \sin n\pi x \Big|_0^{\frac{2}{3}} \right) + \frac{6}{n\pi} \left(\cancel{-\cos n\pi} + \cancel{\cos \frac{2n\pi}{3}} \right) \\ &\qquad - 6 \left(-\cancel{\frac{\cos n\pi}{n\pi}} + \cancel{\frac{2}{3n\pi} \cos \frac{2n\pi}{3}} + \frac{1}{n^2\pi^2} \sin n\pi x \Big|_{\frac{2}{3}}^1 \right) \\ &= \frac{3}{n^2\pi^2} \sin \frac{2n\pi}{3} - 6 \left(\frac{1}{n^2\pi^2} \underbrace{\sin n\pi}_{=0} - \frac{1}{n^2\pi^2} \sin \frac{2n\pi}{3} \right) \\ &= \frac{3}{n^2\pi^2} \sin \frac{2n\pi}{3} + \frac{6}{n^2\pi^2} \sin \frac{2n\pi}{3} \\ &= \frac{9}{n^2\pi^2} \sin \frac{2n\pi}{3} \end{aligned}$$

Thus, we have

$$v(x,t) = \sum_{n=1}^{\infty} \frac{9}{n^2\pi^2} \sin \frac{2n\pi}{3} e^{-n^2\pi^2 t} \sin n\pi x.$$

Therefore,

$$u(x,t) = x + \sum_{n=1}^{\infty} \frac{9}{n^2\pi^2} \sin \frac{2n\pi}{3} e^{-n^2\pi^2 t} \sin n\pi x.$$

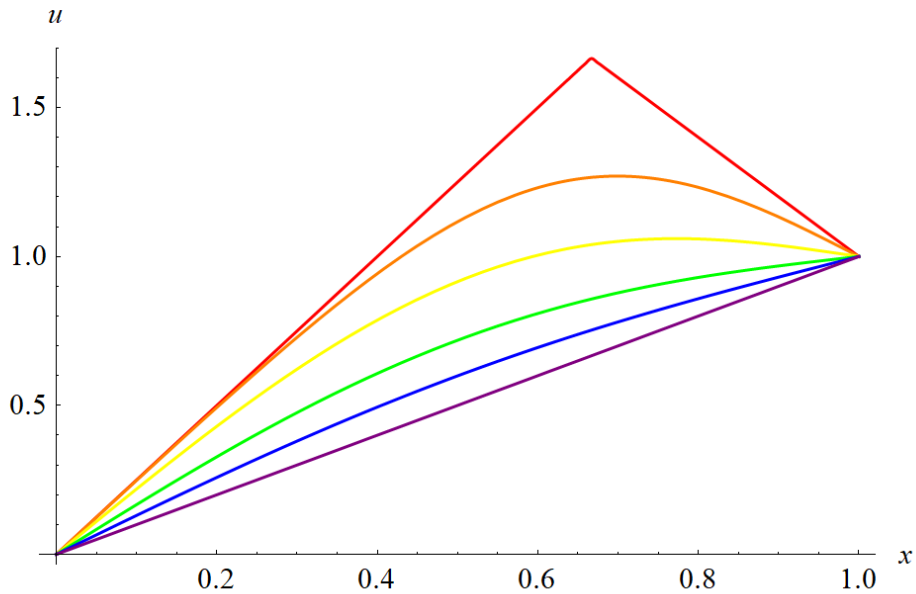

Figure 1: This is a plot of the solution $u(x,t)$ versus x at $t = 0$, $t = 0.025$, $t = 0.065$, $t = 0.13$, $t = 0.21$, and $t = 10$ in red, orange, yellow, green, blue, and purple, respectively.

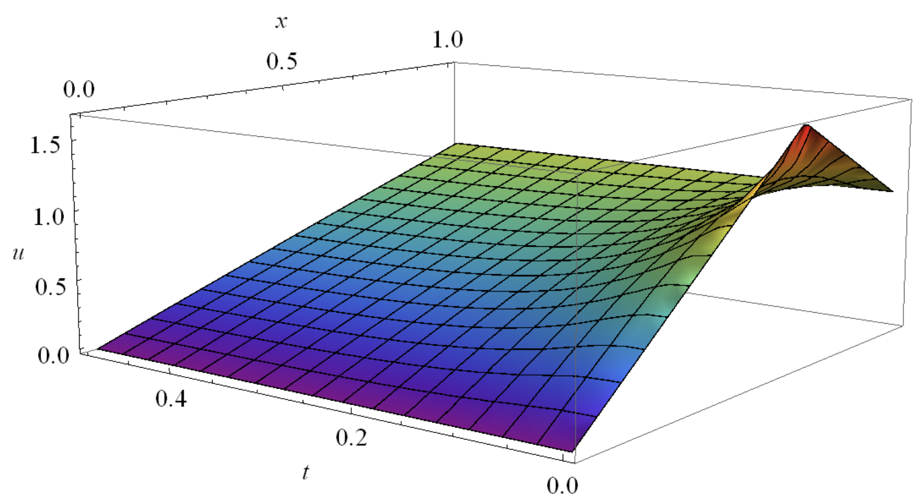

Figure 2: This is a plot of the two-dimensional solution surface in three-dimensional space.