Exercise 10

A string (of tension T and density ρ) with fixed ends at $x = 0$ and $x = l$ is hit by a hammer so that $u(x, 0) = 0$, and $\partial u/\partial t(x, 0) = V$ in $[-\delta + \frac{1}{2}l, \delta + \frac{1}{2}l]$ and $\partial u/\partial t(x, 0) = 0$ elsewhere. Find the solution explicitly in series form. Find the energy

$$E_n(h) = \frac{1}{2}\int_0^l \left[\rho\left(\frac{\partial h}{\partial t}\right)^2 + T\left(\frac{\partial h}{\partial x}\right)^2\right] dx$$

of the nth harmonic $h = h_n$. Conclude that if δ is small (a concentrated blow), each of the first few overtones has almost as much energy as the fundamental. We could say that the tone is saturated with overtones.

Solution

The governing equation of motion for the string is the wave equation with $c^2 = T/\rho$,

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < l.$$

The fact that the string has fixed ends implies there are Dirichlet boundary conditions at $x = 0$ and $x = l$,

$$u(0, t) = 0 \quad \text{and} \quad u(l, t) = 0.$$

The initial conditions are

$$u(x, 0) = 0 \quad \text{and} \quad u_t(x, 0) = g(x) = \begin{cases} 0 & 0 \le x < \frac{l}{2} - \delta \\ V & \frac{l}{2} - \delta \le x \le \frac{l}{2} + \delta \\ 0 & \frac{l}{2} + \delta < x \le l \end{cases}.$$

The PDE and its boundary conditions are linear and homogeneous, so the method of separation of variables can be applied. Assume a product solution of the form $u(x, t) = X(x)T(t)$ and plug it into the PDE

$$u_{tt} = c^2 u_{xx} \quad \to \quad XT'' = c^2 X''T$$

and the boundary conditions.

$$\begin{aligned} u(0, t) = 0 \quad &\to \quad X(0)T(t) = 0 \quad &\to \quad X(0) = 0 \\ u(l, t) = 0 \quad &\to \quad X(l)T(t) = 0 \quad &\to \quad X(l) = 0 \end{aligned}$$

Now separate variables in the PDE: bring all functions of t and constants to the left side and all functions of x to the right side. The final answer would be the same if c^2 were brought to the right side.

$$\frac{T''}{c^2 T} = \frac{X''}{X}$$

The only way a function of t on the left can be equal to a function of x on the right is if both sides are equal to a constant λ.

$$\frac{T''}{c^2 T} = \frac{X''}{X} = \lambda$$

Values of λ for which $X(0) = 0$ and $X(l) = 0$ are satisfied are called the eigenvalues, and the nontrivial functions $X(x)$ associated with them are called the eigenfunctions.

Determination of Positive Eigenvalues: $\lambda = \mu^2$

Assuming λ is positive, the differential equation for X becomes

$$\frac{X''}{X} = \mu^2.$$

Multiply both sides by X.

$$X'' = \mu^2 X$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \mu x + C_2 \sinh \mu x$$

Now use the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} X(0) &= C_1 = 0 \\ X(l) &= C_1 \cosh \mu l + C_2 \sinh \mu l = 0 \end{aligned}$$

The second equation simplifies to $C_2 \sinh \mu l = 0$, which can only be satisfied if $C_2 = 0$. Only the trivial solution $X(x) = 0$ results from considering positive values for λ, so there are no positive eigenvalues.

Determination of the Zero Eigenvalue: $\lambda = 0$

Assuming λ is zero, the differential equation for X becomes

$$\frac{X''}{X} = 0.$$

Multiply both sides by X.

$$X'' = 0$$

The general solution is a linear function.

$$X(x) = C_3 x + C_4$$

Now use the boundary conditions to determine C_3 and C_4.

$$\begin{aligned} X(0) &= C_4 = 0 \\ X(l) &= C_3 l + C_4 = 0 \end{aligned}$$

We find that $C_3 = 0$ and $C_4 = 0$. Only the trivial solution $X(x) = 0$ is obtained, so zero is not an eigenvalue.

Determination of Negative Eigenvalues: $\lambda = -\gamma^2$

Assuming λ is negative, the differential equation for X becomes

$$\frac{X''}{X} = -\gamma^2.$$

Multiply both sides by X.

$$X'' = -\gamma^2 X$$

The general solution can be written in terms of sine and cosine.

$$X(x) = C_5 \cos \gamma x + C_6 \sin \gamma x$$

Apply the boundary conditions here to determine C_5 and C_6.

$$\begin{aligned} X(0) &= C_5 = 0 \\ X(l) &= C_5 \cos \gamma l + C_6 \sin \gamma l = 0 \end{aligned}$$

The second equation reduces to

$$C_6 \sin \gamma l = 0.$$

In order to avoid getting the trivial solution, we insist that $C_6 \neq 0$. The equation for γ is then

$$\begin{aligned} \sin \gamma l &= 0 \\ \gamma l &= n\pi \quad \rightarrow \quad \gamma_n = \frac{n\pi}{l}, \quad n = 1, 2, \ldots. \end{aligned}$$

The eigenfunctions associated with these eigenvalues are

$$X(x) = C_6 \sin \gamma x \quad \rightarrow \quad X_n(x) = \sin \frac{n\pi x}{l}.$$

The ODE for $T(t)$ will now be solved.

$$\frac{T''}{c^2 T} = -\gamma^2$$

Multiply both sides by $c^2 T$.

$$T'' = -c^2 \gamma^2 T$$

The general solution can be written in terms of sine and cosine.

$$T(t) = C_7 \cos c\gamma t + C_8 \sin c\gamma t \quad \rightarrow \quad T_n(t) = C_7 \cos \frac{cn\pi t}{l} + C_8 \sin \frac{cn\pi t}{l}$$

According to the principle of linear superposition, the solution to the PDE for $u(x,t)$ is a linear combination of all products $T_n(t)X_n(x)$ over all the eigenvalues.

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{cn\pi t}{l} + B_n \sin \frac{cn\pi t}{l} \right) \sin \frac{n\pi x}{l}$$

Now we will use the initial conditions to determine the coefficients, A_n and B_n.

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} = 0$$

We can say that $A_n = 0$, so the general solution reduces to

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{cn\pi t}{l} \sin \frac{n\pi x}{l}.$$

Take the derivative with respect to t to use the second initial condition.

$$u_t(x,t) = \sum_{n=1}^{\infty} B_n \frac{cn\pi}{l} \cos \frac{cn\pi t}{l} \sin \frac{n\pi x}{l}$$

Plug in $t = 0$ and use the second initial condition.

$$u_t(x,0) = \sum_{n=1}^{\infty} B_n \frac{cn\pi}{l} \sin \frac{n\pi x}{l} = g(x)$$

To determine $B_n(cn\pi/l)$, multiply both sides by $\sin(m\pi x/l)$, where m is an integer.

$$\sum_{n=1}^{\infty} B_n \frac{cn\pi}{l} \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} = g(x) \sin \frac{m\pi x}{l}$$

Integrate both sides with respect to x over the domain the PDE is defined.

$$\int_0^l \sum_{n=1}^{\infty} B_n \frac{cn\pi}{l} \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l}\, dx = \int_0^l g(x) \sin \frac{m\pi x}{l}\, dx$$

Bring the integral inside the sum on the left side.

$$\sum_{n=1}^{\infty} B_n \frac{cn\pi}{l} \int_0^l \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l}\, dx = \int_0^l g(x) \sin \frac{m\pi x}{l}\, dx$$

If $n \neq m$, then the integral on the left is equal to 0 thanks to the orthogonality of the trigonometric functions. This can be verified with the product-to-sum formula for sine. When $n = m$, the integrand becomes $\sin^2(n\pi x/l)$, and the result of the integral is $l/2$.

$$B_n \frac{cn\pi}{l} \cdot \frac{l}{2} = \int_0^l g(x) \sin \frac{n\pi x}{l}\, dx$$

Solve for B_n.

$$\begin{aligned}
B_n &= \frac{2}{cn\pi} \int_0^l g(x) \sin \frac{n\pi x}{l}\, dx \\
&= \frac{2}{cn\pi} \int_{\frac{l}{2}-\delta}^{\frac{l}{2}+\delta} V \sin \frac{n\pi x}{l}\, dx \\
&= \frac{2V}{cn\pi} \left(-\frac{l}{n\pi}\right) \cos \frac{n\pi x}{l} \bigg|_{\frac{l}{2}-\delta}^{\frac{l}{2}+\delta} \\
&= \frac{2Vl}{cn^2\pi^2} \left\{ \cos\left[\frac{n\pi}{l}\left(\frac{l}{2}-\delta\right)\right] - \cos\left[\frac{n\pi}{l}\left(\frac{l}{2}+\delta\right)\right] \right\} \\
&= \frac{2Vl}{cn^2\pi^2} \left[\cos\left(\frac{n\pi}{2} - \frac{n\pi\delta}{l}\right) - \cos\left(\frac{n\pi}{2} + \frac{n\pi\delta}{l}\right) \right]
\end{aligned}$$

We can use the product-to-sum formula for sine to simplify the expression, which says that

$$\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)].$$

Consequently,

$$\begin{aligned} B_n &= \frac{2Vl}{cn^2\pi^2} \cdot 2\sin\frac{n\pi}{2}\sin\frac{n\pi\delta}{l} \\ &= \frac{4Vl}{cn^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi\delta}{l}. \end{aligned}$$

Now that A_n and B_n are determined, the solution to the PDE is known.

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} B_n \sin\frac{cn\pi t}{l}\sin\frac{n\pi x}{l} \\ &= \sum_{n=1}^{\infty} \frac{4Vl}{cn^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi\delta}{l}\sin\frac{cn\pi t}{l}\sin\frac{n\pi x}{l} \qquad (1) \\ &= \frac{4Vl}{c\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi\delta}{l}\sin\frac{cn\pi t}{l}\sin\frac{n\pi x}{l} \end{aligned}$$

Notice that because of the $\sin(n\pi/2)$ term, the summand is zero for even integers. The answer can thus be simplified (that is, made to converge faster) by summing over the odd integers only. Let $n = 2k-1$ in the sum.

$$u(x,t) = \frac{4Vl}{c\pi^2}\sum_{2k-1=1}^{\infty}\frac{1}{(2k-1)^2}\sin\frac{(2k-1)\pi}{2}\sin\frac{(2k-1)\pi\delta}{l}\sin\frac{c(2k-1)\pi t}{l}\sin\frac{(2k-1)\pi x}{l}$$

Write $\sin[(2k-1)\pi/2]$ as $(-1)^{k+1}$. Also, replace c with $\sqrt{T/\rho}$; the answer should be expressed in terms of the quantities given in the problem statement. Therefore,

$$\boxed{u(x,t) = \frac{4Vl}{\pi^2}\sqrt{\frac{\rho}{T}}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^2}\sin\frac{(2k-1)\pi\delta}{l}\sin\left[\sqrt{\frac{T}{\rho}}\frac{(2k-1)\pi t}{l}\right]\sin\frac{(2k-1)\pi x}{l}.}$$

The nth harmonic $h = h_n$ is the summand in equation (1),

$$h = \frac{4Vl}{cn^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi\delta}{l}\sin\frac{cn\pi t}{l}\sin\frac{n\pi x}{l}.$$

In order to find the energy, substitute this into the provided formula for $E_n(h)$.

$$\begin{aligned} E_n(h) &= \frac{1}{2}\int_0^l \left[\rho\left(\frac{\partial h}{\partial t}\right)^2 + T\left(\frac{\partial h}{\partial x}\right)^2\right] dx \\ &= \frac{1}{2}\int_0^l \left\{\rho\left[\frac{\partial}{\partial t}\left(\frac{4Vl}{cn^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi\delta}{l}\sin\frac{cn\pi t}{l}\sin\frac{n\pi x}{l}\right)\right]^2 \right.\\ &\qquad \left. + T\left[\frac{\partial}{\partial x}\left(\frac{4Vl}{cn^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi\delta}{l}\sin\frac{cn\pi t}{l}\sin\frac{n\pi x}{l}\right)\right]^2\right\} dx \\ &= \frac{1}{2}\int_0^l \left\{\rho\left[\frac{cn\pi}{l}\left(\frac{4Vl}{cn^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi\delta}{l}\cos\frac{cn\pi t}{l}\sin\frac{n\pi x}{l}\right)\right]^2 \right.\\ &\qquad \left. + T\left[\frac{n\pi}{l}\left(\frac{4Vl}{cn^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi\delta}{l}\sin\frac{cn\pi t}{l}\cos\frac{n\pi x}{l}\right)\right]^2\right\} dx \end{aligned}$$

$$\begin{aligned}E_n(h) &= \frac{1}{2}\int_0^l \left[\rho\left(\frac{4V}{n\pi}\sin\frac{n\pi}{2}\sin\frac{n\pi\delta}{l}\cos\frac{cn\pi t}{l}\sin\frac{n\pi x}{l}\right)^2 \right.\\ &\qquad\qquad \left. + T\left(\frac{4V}{cn\pi}\sin\frac{n\pi}{2}\sin\frac{n\pi\delta}{l}\sin\frac{cn\pi t}{l}\cos\frac{n\pi x}{l}\right)^2\right] dx \\ &= \frac{1}{2}\int_0^l \left(\rho\frac{16V^2}{n^2\pi^2}\sin^2\frac{n\pi}{2}\sin^2\frac{n\pi\delta}{l}\cos^2\frac{cn\pi t}{l}\sin^2\frac{n\pi x}{l}\right.\\ &\qquad\qquad \left. + T\frac{16V^2}{c^2n^2\pi^2}\sin^2\frac{n\pi}{2}\sin^2\frac{n\pi\delta}{l}\sin^2\frac{cn\pi t}{l}\cos^2\frac{n\pi x}{l}\right) dx\end{aligned}$$

Replace c^2 with T/ρ.

$$\begin{aligned}&= \frac{1}{2}\int_0^l \left(\rho\frac{16V^2}{n^2\pi^2}\sin^2\frac{n\pi}{2}\sin^2\frac{n\pi\delta}{l}\cos^2\frac{cn\pi t}{l}\sin^2\frac{n\pi x}{l}\right.\\ &\qquad\qquad \left. + \rho\frac{16V^2}{n^2\pi^2}\sin^2\frac{n\pi}{2}\sin^2\frac{n\pi\delta}{l}\sin^2\frac{cn\pi t}{l}\cos^2\frac{n\pi x}{l}\right) dx\end{aligned}$$

Factor the integrand.

$$= \frac{1}{2}\int_0^l \rho\frac{16V^2}{n^2\pi^2}\sin^2\frac{n\pi}{2}\sin^2\frac{n\pi\delta}{l}\left(\cos^2\frac{cn\pi t}{l}\sin^2\frac{n\pi x}{l} + \sin^2\frac{cn\pi t}{l}\cos^2\frac{n\pi x}{l}\right) dx$$

Bring the constants in front of the integral.

$$= \frac{8\rho V^2}{n^2\pi^2}\sin^2\frac{n\pi}{2}\sin^2\frac{n\pi\delta}{l}\int_0^l \left(\cos^2\frac{cn\pi t}{l}\sin^2\frac{n\pi x}{l} + \sin^2\frac{cn\pi t}{l}\cos^2\frac{n\pi x}{l}\right) dx$$

Split up the integral into two and bring the terms with t in front of them.

$$= \frac{8\rho V^2}{n^2\pi^2}\sin^2\frac{n\pi}{2}\sin^2\frac{n\pi\delta}{l}\left(\cos^2\frac{cn\pi t}{l}\int_0^l \sin^2\frac{n\pi x}{l}\,dx + \sin^2\frac{cn\pi t}{l}\int_0^l \cos^2\frac{n\pi x}{l}\,dx\right)$$

The integral of sine squared and cosine squared is $l/2$.

$$= \frac{8\rho V^2}{n^2\pi^2}\sin^2\frac{n\pi}{2}\sin^2\frac{n\pi\delta}{l}\left(\cos^2\frac{cn\pi t}{l}\cdot\frac{l}{2} + \sin^2\frac{cn\pi t}{l}\cdot\frac{l}{2}\right)$$

Factor $l/2$.

$$= \frac{4\rho V^2 l}{n^2\pi^2}\sin^2\frac{n\pi}{2}\sin^2\frac{n\pi\delta}{l}\underbrace{\left(\cos^2\frac{cn\pi t}{l} + \sin^2\frac{cn\pi t}{l}\right)}_{=1}$$

Use the fact that $\cos^2 z + \sin^2 z = 1$. Therefore, the energy of the nth harmonic is

$$\boxed{E_n = \frac{4\rho V^2 l}{n^2\pi^2}\sin^2\frac{n\pi}{2}\sin^2\frac{n\pi\delta}{l} = \begin{cases}0 & n \text{ even} \\ \dfrac{4\rho V^2 l}{n^2\pi^2}\sin^2\dfrac{n\pi\delta}{l} & n \text{ odd}\end{cases}}.$$

Recall that the Taylor series expansion of the sine function is

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots,$$

so

$$\sin\frac{n\pi\delta}{l} = \frac{n\pi\delta}{l} - \frac{1}{3!}\left(\frac{n\pi\delta}{l}\right)^3 + \frac{1}{5!}\left(\frac{n\pi\delta}{l}\right)^5 - \cdots.$$

If δ is extremely small ($\delta \ll 1$), then the sine function can be reasonably approximated by the first term because the higher orders of δ are negligible in comparison.

$$\sin\frac{n\pi\delta}{l} \approx \frac{n\pi\delta}{l}$$

In this case, the formula for the energy can be simplified.

$$\begin{aligned} E_n &\approx \frac{4\rho V^2 l}{n^2\pi^2}\sin^2\frac{n\pi}{2}\left(\frac{n\pi\delta}{l}\right)^2 = \begin{cases} 0 & n \text{ even} \\ \frac{4\rho V^2 l}{n^2\pi^2}\left(\frac{n\pi\delta}{l}\right)^2 & n \text{ odd} \end{cases} \\ &\approx \frac{4\rho V^2\delta^2}{l}\sin^2\frac{n\pi}{2} = \begin{cases} 0 & n \text{ even} \\ \frac{4\rho V^2\delta^2}{l} & n \text{ odd} \end{cases} \end{aligned}$$

Whether $n = 1$ or $n = 3$ or $n = 5$, the energy is the same. Let's now suppose that δ is a little bit bigger and use the first two terms of the Taylor series. Then

$$\begin{aligned} E_n &\approx \frac{4\rho V^2 l}{n^2\pi^2}\sin^2\frac{n\pi}{2}\left[\frac{n\pi\delta}{l} - \frac{1}{3!}\left(\frac{n\pi\delta}{l}\right)^3\right]^2 = \begin{cases} 0 & n \text{ even} \\ \frac{4\rho V^2 l}{n^2\pi^2}\left[\frac{n\pi\delta}{l} - \frac{1}{3!}\left(\frac{n\pi\delta}{l}\right)^3\right]^2 & n \text{ odd} \end{cases} \\ &\approx \frac{4\rho V^2\delta^2}{l}\sin^2\frac{n\pi}{2}\left[1 - \frac{1}{3!}\left(\frac{n\pi\delta}{l}\right)^2\right]^2 = \begin{cases} 0 & n \text{ even} \\ \frac{4\rho V^2\delta^2}{l}\left[1 - \frac{1}{3!}\left(\frac{n\pi\delta}{l}\right)^2\right]^2 & n \text{ odd} \end{cases}. \end{aligned}$$

Notice that when $n = 1$ the energy is maximum and that when $n = 3$ or $n = 5$ the energy is slightly less. Therefore, if δ is small, then the first few overtones ($n = 3$ and $n = 5$) will have roughly the same energy as the fundamental ($n = 1$).

TYPO: This answer is in disagreement with the one at the back of the book,

$$E_n = [4l\rho V^2/(n\pi)^2]\sin^2(n\pi\delta/l) \sim 4\rho V^2 l^{-1}\delta^2 \quad \text{for fixed even } n \text{ and small } \delta.$$

I believe it should say, "for fixed odd n and small δ."