Exercise 11

On a string with fixed ends, show that if the center of a hammer blow is exactly at a node of the nth harmonic (a place where the nth eigenfunction vanishes), the nth overtone is absent from the solution.

Solution

The displacement of a string with tension T, mass density ρ, and length l satisfies the wave equation with $c^2 = T/\rho$,

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < l.$$

The fact that the string has fixed ends implies there are Dirichlet boundary conditions at $x = 0$ and $x = l$,

$$u(0,t) = 0 \quad \text{and} \quad u(l,t) = 0.$$

The initial conditions for a hammer blow are

$$u(x,0) = 0 \quad \text{and} \quad u_t(x,0) = g(x) = \begin{cases} 0 & 0 \le x < P - \delta \\ V & P - \delta \le x \le P + \delta \\ 0 & P + \delta < x \le l \end{cases},$$

where P is the point on the string where the center of the hammer's head strikes and δ is the head's radius. The PDE and its boundary conditions are linear and homogeneous, so the method of separation of variables can be applied. Assume a product solution of the form $u(x,t) = X(x)T(t)$ and plug it into the PDE

$$u_{tt} = c^2 u_{xx} \quad \rightarrow \quad XT'' = c^2 X'' T$$

and the boundary conditions.

$$\begin{aligned} u(0,t) = 0 \quad &\rightarrow \quad X(0)T(t) = 0 \quad &\rightarrow \quad X(0) = 0 \\ u(l,t) = 0 \quad &\rightarrow \quad X(l)T(t) = 0 \quad &\rightarrow \quad X(l) = 0 \end{aligned}$$

Now separate variables in the PDE: bring all functions of t and constants to the left side and all functions of x to the right side. The final answer would be the same if c^2 were brought to the right side.

$$\frac{T''}{c^2 T} = \frac{X''}{X}$$

The only way a function of t on the left can be equal to a function of x on the right is if both sides are equal to a constant λ.

$$\frac{T''}{c^2 T} = \frac{X''}{X} = \lambda$$

Values of λ for which $X(0) = 0$ and $X(l) = 0$ are satisfied are called the eigenvalues, and the nontrivial functions $X(x)$ associated with them are called the eigenfunctions.

Determination of Positive Eigenvalues: $\lambda = \mu^2$

Assuming λ is positive, the differential equation for X becomes

$$\frac{X''}{X} = \mu^2.$$

Multiply both sides by X.

$$X'' = \mu^2 X$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \mu x + C_2 \sinh \mu x$$

Now use the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} X(0) &= C_1 = 0 \\ X(l) &= C_1 \cosh \mu l + C_2 \sinh \mu l = 0 \end{aligned}$$

The second equation simplifies to $C_2 \sinh \mu l = 0$, which can only be satisfied if $C_2 = 0$. Only the trivial solution $X(x) = 0$ results from considering positive values for λ, so there are no positive eigenvalues.

Determination of the Zero Eigenvalue: $\lambda = 0$

Assuming λ is zero, the differential equation for X becomes

$$\frac{X''}{X} = 0.$$

Multiply both sides by X.

$$X'' = 0$$

The general solution is a linear function.

$$X(x) = C_3 x + C_4$$

Now use the boundary conditions to determine C_3 and C_4.

$$\begin{aligned} X(0) &= C_4 = 0 \\ X(l) &= C_3 l + C_4 = 0 \end{aligned}$$

We find that $C_3 = 0$ and $C_4 = 0$. Only the trivial solution $X(x) = 0$ is obtained, so zero is not an eigenvalue.

Determination of Negative Eigenvalues: $\lambda = -\gamma^2$

Assuming λ is negative, the differential equation for X becomes

$$\frac{X''}{X} = -\gamma^2.$$

Multiply both sides by X.

$$X'' = -\gamma^2 X$$

The general solution can be written in terms of sine and cosine.

$$X(x) = C_5 \cos\gamma x + C_6 \sin\gamma x$$

Apply the boundary conditions here to determine C_5 and C_6.

$$X(0) = C_5 = 0$$
$$X(l) = C_5 \cos\gamma l + C_6 \sin\gamma l = 0$$

The second equation reduces to

$$C_6 \sin\gamma l = 0.$$

In order to avoid getting the trivial solution, we insist that $C_6 \neq 0$. The equation for γ is then

$$\sin\gamma l = 0$$
$$\gamma l = n\pi \quad \rightarrow \quad \gamma_n = \frac{n\pi}{l}, \quad n = 1, 2, \ldots.$$

The eigenfunctions associated with these eigenvalues are

$$X(x) = C_6 \sin\gamma x \quad \rightarrow \quad X_n(x) = \sin\frac{n\pi x}{l}.$$

The ODE for $T(t)$ will now be solved.

$$\frac{T''}{c^2 T} = -\gamma^2$$

Multiply both sides by $c^2 T$.

$$T'' = -c^2\gamma^2 T$$

The general solution can be written in terms of sine and cosine.

$$T(t) = C_7 \cos c\gamma t + C_8 \sin c\gamma t \quad \rightarrow \quad T_n(t) = C_7 \cos\frac{cn\pi t}{l} + C_8 \sin\frac{cn\pi t}{l}$$

According to the principle of linear superposition, the solution to the PDE for $u(x,t)$ is a linear combination of all products $T_n(t)X_n(x)$ over all the eigenvalues.

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n \cos\frac{cn\pi t}{l} + B_n \sin\frac{cn\pi t}{l}\right)\sin\frac{n\pi x}{l}$$

Now we will use the initial conditions to determine the coefficients, A_n and B_n.

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} = 0$$

We can say that $A_n = 0$, so the general solution reduces to

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{cn\pi t}{l}\sin\frac{n\pi x}{l}.$$

Take the derivative with respect to t to use the second initial condition.

$$u_t(x,t) = \sum_{n=1}^{\infty} B_n \frac{cn\pi}{l}\cos\frac{cn\pi t}{l}\sin\frac{n\pi x}{l}$$

Plug in $t = 0$ and use the second initial condition.

$$u_t(x,0) = \sum_{n=1}^{\infty} B_n \frac{cn\pi}{l} \sin\frac{n\pi x}{l} = g(x)$$

To determine $B_n(cn\pi/l)$, multiply both sides by $\sin(m\pi x/l)$, where m is an integer.

$$\sum_{n=1}^{\infty} B_n \frac{cn\pi}{l} \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l} = g(x)\sin\frac{m\pi x}{l}$$

Integrate both sides with respect to x over the domain the PDE is defined.

$$\int_0^l \sum_{n=1}^{\infty} B_n \frac{cn\pi}{l} \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l}\,dx = \int_0^l g(x)\sin\frac{m\pi x}{l}\,dx$$

Bring the integral inside the sum on the left side.

$$\sum_{n=1}^{\infty} B_n \frac{cn\pi}{l} \int_0^l \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l}\,dx = \int_0^l g(x)\sin\frac{m\pi x}{l}\,dx$$

If $n \neq m$, then the integral on the left is equal to 0 thanks to the orthogonality of the trigonometric functions. This can be verified with the product-to-sum formula for sine. When $n = m$, the integrand becomes $\sin^2(n\pi x/l)$, and the result of the integral is $l/2$.

$$B_n \frac{cn\pi}{l} \cdot \frac{l}{2} = \int_0^l g(x)\sin\frac{n\pi x}{l}\,dx$$

Solve for B_n.

$$\begin{aligned}
B_n &= \frac{2}{cn\pi}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx \\
&= \frac{2}{cn\pi}\int_{P-\delta}^{P+\delta} V\sin\frac{n\pi x}{l}\,dx \\
&= \frac{2V}{cn\pi}\left(-\frac{l}{n\pi}\right)\cos\frac{n\pi x}{l}\bigg|_{P-\delta}^{P+\delta} \\
&= \frac{2Vl}{cn^2\pi^2}\left\{\cos\left[\frac{n\pi}{l}(P-\delta)\right] - \cos\left[\frac{n\pi}{l}(P+\delta)\right]\right\} \\
&= \frac{2Vl}{cn^2\pi^2}\left[\cos\left(\frac{n\pi P}{l} - \frac{n\pi\delta}{l}\right) - \cos\left(\frac{n\pi P}{l} + \frac{n\pi\delta}{l}\right)\right]
\end{aligned}$$

We can use the product-to-sum formula for sine to simplify the expression, which says that

$$\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)].$$

Consequently,

$$\begin{aligned}
B_n &= \frac{2Vl}{cn^2\pi^2} \cdot 2\sin\frac{n\pi P}{l}\sin\frac{n\pi\delta}{l} \\
&= \frac{4Vl}{cn^2\pi^2}\sin\frac{n\pi P}{l}\sin\frac{n\pi\delta}{l}.
\end{aligned}$$

Now that A_n and B_n are determined, the solution to the PDE is known.

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} B_n \sin\frac{cn\pi t}{l} \sin\frac{n\pi x}{l} \\ &= \sum_{n=1}^{\infty} \frac{4Vl}{cn^2\pi^2} \sin\frac{n\pi P}{l} \sin\frac{n\pi\delta}{l} \sin\frac{cn\pi t}{l} \sin\frac{n\pi x}{l} \end{aligned}$$

The nth harmonic h_n is the summand.

$$h_n = \frac{4Vl}{cn^2\pi^2} \sin\frac{n\pi P}{l} \sin\frac{n\pi\delta}{l} \sin\frac{cn\pi t}{l} \sin\frac{n\pi x}{l}$$

Nodes of the nth harmonic occur on the string where

$$\sin\frac{n\pi x}{l} = 0.$$

This implies that

$$\frac{n\pi x}{l} = m\pi,$$

where m is an integer. Solving this equation for x gives the locations of the nodes on the string.

$$x_m = \frac{m}{n} l$$

For a given value of n, $\sin(n\pi x/l)$ has $n-1$ zeros in the interval $0 < x < l$. x_m represents the mth zero in this interval; m itself takes on the values $0 < m < n$. Suppose now that the center of the hammer blow is at one of these nodes.

$$P = x_m = \frac{m}{n} l$$

The first overtone to the nth harmonic is the $(n+1)$th harmonic, the second overtone to the nth harmonic is the $(n+2)$th harmonic, and the nth overtone to the nth harmonic is the $(2n)$th harmonic.

$$\begin{aligned} h_{2n} &= \frac{4Vl}{c(2n)^2\pi^2} \sin\frac{(2n)\pi P}{l} \sin\frac{(2n)\pi\delta}{l} \sin\frac{c(2n)\pi t}{l} \sin\frac{(2n)\pi x}{l} \\ &= \frac{Vl}{cn^2\pi^2} \sin\frac{2n\pi P}{l} \sin\frac{2n\pi\delta}{l} \sin\frac{2cn\pi t}{l} \sin\frac{2n\pi x}{l} \end{aligned}$$

Substitute $P = ml/n$ here.

$$\begin{aligned} &= \frac{Vl}{cn^2\pi^2} \underbrace{\sin(2m\pi)}_{=\,0} \sin\frac{2n\pi\delta}{l} \sin\frac{2cn\pi t}{l} \sin\frac{2n\pi x}{l} \\ &= 0 \end{aligned}$$

Therefore, if the center of a hammer blow is exactly at a node of the nth harmonic, the nth overtone is absent from the solution.