Exercise 10

(a) Let $\phi(x)$ be a continuous function on $(0, l)$. Under what conditions is its *odd* extension also a continuous function?

(b) Let $\phi(x)$ be a differentiable function on $(0, l)$. Under what conditions is its *odd* extension also a differentiable function?

(c) Same as part (a) for the *even* extension.

(d) Same as part (b) for the *even* extension.

Solution

The odd extension of $\phi(x)$ is defined as

$$\phi_{\text{odd}} = \begin{cases} \phi(x) & 0 < x < l \\ -\phi(-x) & -l < x < 0 \\ 0 & x = 0 \end{cases},$$

and the even extension of $\phi(x)$ is defined as

$$\phi_{\text{even}} = \begin{cases} \phi(x) & 0 < x < l \\ \phi(-x) & -l < x < 0 \end{cases}.$$

Part (a)

The odd extension is continuous as long as $\phi(x)$ passes through the origin.

$$\lim_{x \to 0^+} \phi(x) = 0.$$

Part (b)

The odd extension is differentiable as long as $\phi(x)$ passes through the origin.

$$\lim_{x \to 0^+} \phi(x) = 0.$$

Part (c)

No conditions on $\phi(x)$ are necessary.

Part (d)

For the even extension to be differentiable, the slope at $x = 0$ must be zero.

$$\lim_{x \to 0^+} \phi'(x) = 0$$