Exercise 12

Repeat Exercise 11 for $\cosh x$. (*Hint:* Use the preceding result.)

Solution

Since $\cosh x$ can be written in terms of the exponential function as

$$\cosh x = \frac{e^x + e^{-x}}{2},$$

the results of the previous exercise can be used to find the complex and real Fourier series of $\cosh x$.

The Complex Fourier Series

From Exercise 11 we have

$$e^x = \sum_{n=-\infty}^{\infty} (-1)^n \frac{l + in\pi}{l^2 + n^2\pi^2} (\sinh l) e^{in\pi x/l}.$$

Plugging in $-x$ for x, we can get the complex Fourier series for e^{-x}.

$$e^{-x} = \sum_{n=-\infty}^{\infty} (-1)^n \frac{l + in\pi}{l^2 + n^2\pi^2} (\sinh l) e^{-in\pi x/l}$$

This is actually not a legitimate Fourier series, though, because the exponent of e has to be $+in\pi x/l$. Thus, we substitute $n = -k$ in the series to make it so.

$$= \sum_{-k=-\infty}^{\infty} (-1)^{-k} \frac{l + i(-k)\pi}{l^2 + (-k)^2\pi^2} (\sinh l) e^{-i(-k)\pi x/l}$$

k essentially runs from $-\infty$ to ∞. Also, $(-1)^{-k} = (-1)^k$.

$$= \sum_{k=-\infty}^{\infty} (-1)^k \frac{l - ik\pi}{l^2 + k^2\pi^2} (\sinh l) e^{ik\pi x/l}$$

Since k is a dummy index, we can replace it with n.

$$= \sum_{n=-\infty}^{\infty} (-1)^n \frac{l - in\pi}{l^2 + n^2\pi^2} (\sinh l) e^{in\pi x/l}$$

So then

$$\begin{aligned} \cosh x &= \frac{1}{2}(e^x + e^{-x}) \\ &= \frac{1}{2}\left[\sum_{n=-\infty}^{\infty} (-1)^n \frac{l + in\pi}{l^2 + n^2\pi^2} (\sinh l) e^{in\pi x/l} + \sum_{n=-\infty}^{\infty} (-1)^n \frac{l - in\pi}{l^2 + n^2\pi^2} (\sinh l) e^{in\pi x/l}\right]. \end{aligned}$$

Combine the sums.

$$\cosh x = \frac{1}{2}\sum_{n=-\infty}^{\infty}\left[(-1)^n \frac{l+in\pi}{l^2+n^2\pi^2}(\sinh l)e^{in\pi x/l} + (-1)^n \frac{l-in\pi}{l^2+n^2\pi^2}(\sinh l)e^{in\pi x/l}\right]$$

Add the terms in the summand.

$$= \frac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^n \frac{2l}{l^2+n^2\pi^2}(\sinh l)e^{in\pi x/l}$$

Therefore, the complex Fourier series of $\cosh x$ on $(-l, l)$ is

$$\cosh x = \sum_{n=-\infty}^{\infty}(-1)^n \frac{l}{l^2+n^2\pi^2}(\sinh l)e^{in\pi x/l}.$$

The Real Fourier Series

To obtain the real Fourier series, split up the sum.

$$\cosh x = \sum_{n=-\infty}^{-1}(-1)^n \frac{l}{l^2+n^2\pi^2}(\sinh l)e^{in\pi x/l} + \frac{\sinh l}{l} + \sum_{n=1}^{\infty}(-1)^n \frac{l}{l^2+n^2\pi^2}(\sinh l)e^{in\pi x/l}$$

Substitute $n = -k$ in the first sum and $n = k$ in the last sum.

$$= \sum_{-k=-\infty}^{-1}(-1)^{-k} \frac{l}{l^2+(-k)^2\pi^2}(\sinh l)e^{i(-k)\pi x/l} + \frac{\sinh l}{l} + \sum_{k=1}^{\infty}(-1)^k \frac{l}{l^2+k^2\pi^2}(\sinh l)e^{ik\pi x/l}$$

k essentially runs from 1 to ∞ in the first sum. Also, $(-1)^{-k} = (-1)^k$.

$$= \sum_{k=1}^{\infty}(-1)^k \frac{l}{l^2+k^2\pi^2}(\sinh l)e^{-ik\pi x/l} + \frac{\sinh l}{l} + \sum_{k=1}^{\infty}(-1)^k \frac{l}{l^2+k^2\pi^2}(\sinh l)e^{ik\pi x/l}$$

Combine the two sums.

$$= \frac{\sinh l}{l} + \sum_{k=1}^{\infty}\left[(-1)^k \frac{l}{l^2+k^2\pi^2}(\sinh l)e^{-ik\pi x/l} + (-1)^k \frac{l}{l^2+k^2\pi^2}(\sinh l)e^{ik\pi x/l}\right]$$

Factor the summand.

$$= \frac{\sinh l}{l} + \sum_{k=1}^{\infty}(-1)^k \frac{l}{l^2+k^2\pi^2}(\sinh l)(e^{-ik\pi x/l} + e^{ik\pi x/l})$$

Use Euler's formula to write the exponential functions in terms of sine and cosine.

$$= \frac{\sinh l}{l} + \sum_{k=1}^{\infty}(-1)^k \frac{l}{l^2+k^2\pi^2}(\sinh l)\left[\left(\cos\frac{k\pi x}{l} - \cancel{i\sin\frac{k\pi x}{l}}\right) + \left(\cos\frac{k\pi x}{l} + \cancel{i\sin\frac{k\pi x}{l}}\right)\right]$$

Simplify the result.

$$= \frac{\sinh l}{l} + \sum_{k=1}^{\infty}(-1)^k \frac{2l}{l^2+k^2\pi^2}(\sinh l)\cos\frac{k\pi x}{l}$$

Bring the constants in front of the sum.

$$\cosh x = \frac{\sinh l}{l} + (2l \sinh l) \sum_{k=1}^{\infty} \frac{(-1)^k}{l^2 + k^2\pi^2} \cos \frac{k\pi x}{l}$$

Replacing the dummy index k with n, therefore, the real Fourier series for $\cosh x$ on $(-l, l)$ is

$$\cosh x = \frac{\sinh l}{l} + (2l \sinh l) \sum_{n=1}^{\infty} \frac{(-1)^n}{l^2 + n^2\pi^2} \cos \frac{n\pi x}{l}.$$

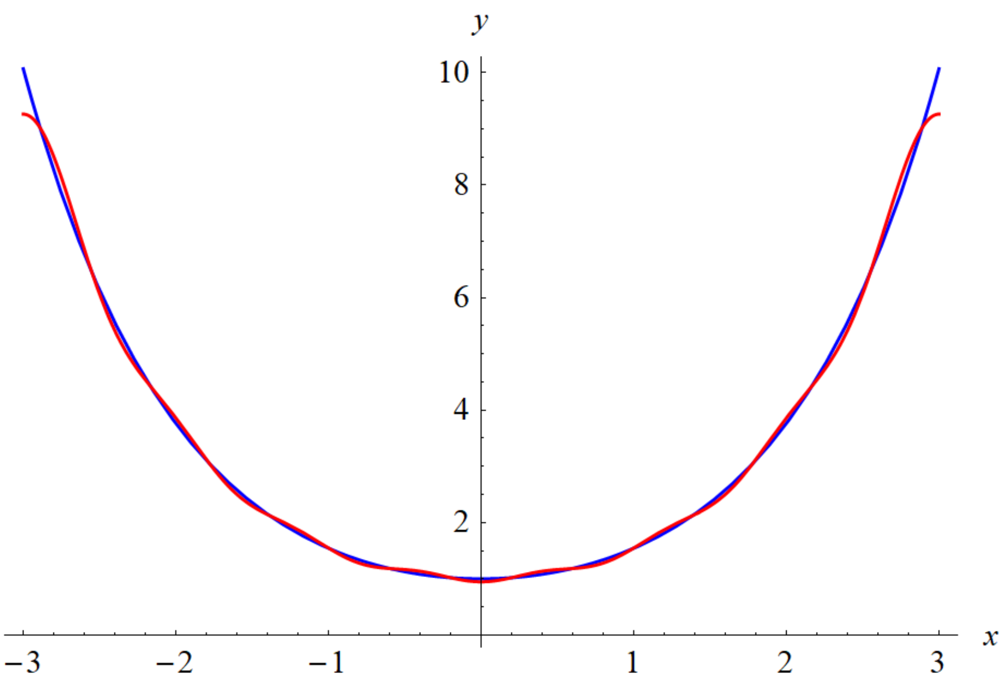

Figure 1: This is a sample plot of $y = \cosh x$ on $(-3, 3)$ in blue. An approximation to the Fourier series of $\cosh x$ is plotted in red, where only the first 7 terms in the infinite series have been used.