Exercise 13

Repeat Exercise 11 for $\sin x$. Assume that l is not an integer multiple of π. (*Hint:* First find the series for e^{ix}.)

Solution

The Complex Fourier Series

The complex Fourier series of $\sin x$ is

$$\sin x = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}.$$

To determine the coefficients c_n, multiply both sides by $e^{-im\pi x/l}$, where m is an integer. We assume that $n \neq m$ for now.

$$(\sin x)e^{-im\pi x/l} = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l} e^{-im\pi x/l}$$

Integrate both sides with respect to x over the interval $(-l, l)$.

$$\int_{-l}^{l} (\sin x)e^{-im\pi x/l}\, dx = \int_{-l}^{l} \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l} e^{-im\pi x/l}\, dx$$

Write the sine function in terms of exponential functions.

$$\int_{-l}^{l} \frac{e^{ix} - e^{-ix}}{2i} e^{-im\pi x/l}\, dx = \int_{-l}^{l} \sum_{n=-\infty}^{\infty} c_n e^{i(n-m)\pi x/l}\, dx$$

Bring the constants in front of the integrals.

$$\frac{1}{2i} \int_{-l}^{l} (e^{ix} - e^{-ix}) e^{-im\pi x/l}\, dx = \sum_{n=-\infty}^{\infty} c_n \int_{-l}^{l} e^{i(n-m)\pi x/l}\, dx$$

Multiply both sides by $2i$.

$$\int_{-l}^{l} (e^{ix} e^{-im\pi x/l} - e^{-ix} e^{-im\pi x/l})\, dx = 2i \sum_{n=-\infty}^{\infty} c_n \int_{-l}^{l} e^{i(n-m)\pi x/l}\, dx$$

Split up the integral on the left side into two and combine the exponential functions.

$$\int_{-l}^{l} e^{ix(1-m\pi/l)}\, dx - \int_{-l}^{l} e^{-ix(1+m\pi/l)}\, dx = 2i \sum_{n=-\infty}^{\infty} c_n \int_{-l}^{l} e^{i(n-m)\pi x/l}\, dx$$

Evaluate the integrals.

$$\frac{1}{i(1-m\pi/l)} e^{ix(1-m\pi/l)} \bigg|_{-l}^{l} - \frac{1}{-i(1+m\pi/l)} e^{-ix(1+m\pi/l)} \bigg|_{-l}^{l} = 2i \sum_{n=-\infty}^{\infty} c_n \cdot \frac{1}{i(n-m)\pi/l} e^{i(n-m)\pi x/l} \bigg|_{-l}^{l}$$

Multiply both sides by i/l.

$$\frac{1}{l-m\pi}e^{ix(1-m\pi/l)}\bigg|_{-l}^{l}+\frac{1}{l+m\pi}e^{-ix(1+m\pi/l)}\bigg|_{-l}^{l}=-\frac{2}{l}\sum_{n=-\infty}^{\infty}c_n\cdot\frac{1}{i(n-m)\pi/l}e^{i(n-m)\pi x/l}\bigg|_{-l}^{l}$$

Multiply both sides by the least common denominator $(l-m\pi)(l+m\pi)=l^2-m^2\pi^2$.

$$\begin{aligned}(l+m\pi)e^{ix(1-m\pi/l)}\bigg|_{-l}^{l}&+(l-m\pi)e^{-ix(1+m\pi/l)}\bigg|_{-l}^{l}\\ &=-\frac{2}{l}(l^2-m^2\pi^2)\sum_{n=-\infty}^{\infty}c_n\cdot\frac{1}{i(n-m)\pi/l}e^{i(n-m)\pi x/l}\bigg|_{-l}^{l}\end{aligned}$$

Now plug in the limits.

$$\begin{aligned}(l+m\pi)[e^{i(l-m\pi)}-e^{-i(l-m\pi)}]&+(l-m\pi)[e^{-i(l+m\pi)}-e^{i(l+m\pi)}]\\ &=-\frac{2}{l}(l^2-m^2\pi^2)\sum_{n=-\infty}^{\infty}c_n\cdot\frac{1}{i(n-m)\pi/l}[e^{i(n-m)\pi}-e^{-i(n-m)\pi}]\end{aligned}$$

Use Euler's formula to write each exponential function in terms of sine and cosine.

$$\begin{aligned}(l+m\pi)&\{[\cos(l-m\pi)+i\sin(l-m\pi)]-[\cos(l-m\pi)-i\sin(l-m\pi)]\}\\ &+(l-m\pi)\{[\cos(l+m\pi)-i\sin(l+m\pi)]-[\cos(l+m\pi)+i\sin(l+m\pi)]\}\\ &=-\frac{2}{l}(l^2-m^2\pi^2)\sum_{n=-\infty}^{\infty}c_n\cdot\frac{1}{i(n-m)\pi/l}\{\cancel{\cos[(n-m)\pi]}+i\underbrace{\sin[(n-m)\pi]}_{=0}\\ &\qquad-\cancel{\cos[(n-m)\pi]}+i\underbrace{\sin[(n-m)\pi]}_{=0}\}\end{aligned}$$

We find that every term in the infinite series is zero if $n\neq m$ as a result of the integral. The $n=m$ term is all that remains on the right side.

$$\begin{aligned}(l+n\pi)&\{[\cos(l-n\pi)+i\sin(l-n\pi)]-[\cos(l-n\pi)-i\sin(l-n\pi)]\}\\ &+(l-n\pi)\{[\cos(l+n\pi)-i\sin(l+n\pi)]-[\cos(l+n\pi)+i\sin(l+n\pi)]\}\\ &=-\frac{2}{l}(l^2-n^2\pi^2)c_n\int_{-l}^{l}e^0\,dx\end{aligned}$$

Simplify the left side and evaluate the integral on the right side.

$$(l+n\pi)[2i\sin(l-n\pi)]+(l-n\pi)[-2i\sin(l+n\pi)]=-\frac{2}{l}(l^2-n^2\pi^2)c_n\cdot 2l$$

Divide both sides by $2i$.

$$(l+n\pi)\sin(l-n\pi)-(l-n\pi)\sin(l+n\pi)=-\frac{2}{i}(l^2-n^2\pi^2)c_n$$

Expand the sines.

$$(l+n\pi)(\sin l\underbrace{\cos n\pi}_{=(-1)^n}-\cos l\underbrace{\sin n\pi}_{=0})-(l-n\pi)(\sin l\underbrace{\cos n\pi}_{=(-1)^n}+\cos l\underbrace{\sin n\pi}_{=0})=-\frac{2}{i}(l^2-n^2\pi^2)c_n$$

Simplify the left side.

$$(l+n\pi)(-1)^n \sin l - (l-n\pi)(-1)^n \sin l = -\frac{2}{i}(l^2 - n^2\pi^2)c_n$$

Combine like-terms.

$$2n\pi(-1)^n \sin l = -\frac{2}{i}(l^2 - n^2\pi^2)c_n$$

Solving for c_n, we have

$$c_n = \frac{in\pi(-1)^{n+1}\sin l}{l^2 - n^2\pi^2}.$$

Therefore, the complex Fourier series of $\sin x$ on $(-l, l)$ is

$$\sin x = \sum_{n=-\infty}^{\infty} \frac{in\pi(-1)^{n+1}\sin l}{l^2 - n^2\pi^2} e^{in\pi x/l}.$$

<u>The Real Fourier Series</u>

To obtain the real Fourier series, split up the sum.

$$\sin x = \sum_{n=-\infty}^{-1} \frac{in\pi(-1)^{n+1}\sin l}{l^2 - n^2\pi^2} e^{in\pi x/l} + 0 + \sum_{n=1}^{\infty} \frac{in\pi(-1)^{n+1}\sin l}{l^2 - n^2\pi^2} e^{in\pi x/l}$$

Substitute $n = -k$ in the first sum and $n = k$ in the last sum.

$$= \sum_{-k=-\infty}^{-1} \frac{i(-k)\pi(-1)^{-k+1}\sin l}{l^2 - (-k)^2\pi^2} e^{i(-k)\pi x/l} + \sum_{k=1}^{\infty} \frac{ik\pi(-1)^{k+1}\sin l}{l^2 - k^2\pi^2} e^{ik\pi x/l}$$

k essentially runs from 1 to ∞ in the first sum. Also, $(-1)^{-k+1} = (-1)^{k+1}$.

$$= -\sum_{k=1}^{\infty} \frac{ik\pi(-1)^{k+1}\sin l}{l^2 - k^2\pi^2} e^{-ik\pi x/l} + \sum_{k=1}^{\infty} \frac{ik\pi(-1)^{k+1}\sin l}{l^2 - k^2\pi^2} e^{ik\pi x/l}$$

Combine the two sums.

$$= \sum_{k=1}^{\infty} \left[-\frac{ik\pi(-1)^{k+1}\sin l}{l^2 - k^2\pi^2} e^{-ik\pi x/l} + \frac{ik\pi(-1)^{k+1}\sin l}{l^2 - k^2\pi^2} e^{ik\pi x/l} \right]$$

Factor the summand.

$$= \sum_{k=1}^{\infty} \frac{ik\pi(-1)^{k+1}\sin l}{l^2 - k^2\pi^2} (-e^{-ik\pi x/l} + e^{ik\pi x/l})$$

Use Euler's formula to write the exponential functions in terms of sine and cosine.

$$= \sum_{k=1}^{\infty} \frac{ik\pi(-1)^{k+1}\sin l}{l^2 - k^2\pi^2} \left[-\left(\cancel{\cos\frac{k\pi x}{l}} - i\sin\frac{k\pi x}{l} \right) + \left(\cancel{\cos\frac{k\pi x}{l}} + i\sin\frac{k\pi x}{l} \right) \right]$$

Simplify the result.

$$\begin{aligned}\sin x &= \sum_{k=1}^{\infty} \frac{ik\pi(-1)(-1)^k \sin l}{l^2 - k^2\pi^2} \left(2i \sin \frac{k\pi x}{l}\right) \\ &= \sum_{k=1}^{\infty} \frac{2k\pi(-1)^k \sin l}{l^2 - k^2\pi^2} \sin \frac{k\pi x}{l} \\ &= (2\pi \sin l) \sum_{k=1}^{\infty} \frac{k(-1)^k}{l^2 - k^2\pi^2} \sin \frac{k\pi x}{l}\end{aligned}$$

Replacing the dummy index k with n, therefore, the real Fourier series for $\sin x$ on $(-l, l)$ is

$$\sin x = (2\pi \sin l) \sum_{n=1}^{\infty} \frac{n(-1)^n}{l^2 - n^2\pi^2} \sin \frac{n\pi x}{l}.$$

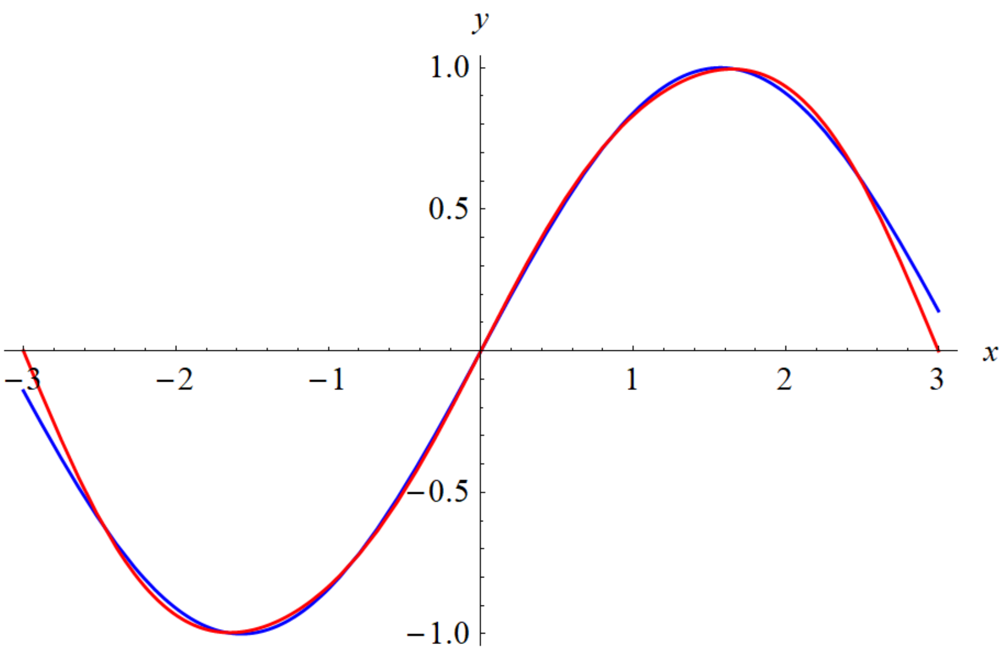

Figure 1: This is a sample plot of $y = \sin x$ on $(-3, 3)$ in blue. An approximation to the Fourier series of $\sin x$ is plotted in red, where only the first 3 terms in the infinite series have been used.