Exercise 17

Show that a complex-valued function $f(x)$ is real-valued if and only if its complex Fourier coefficients satisfy $c_n = \overline{c_{-n}}$, where $\overline{}$ denotes the complex conjugate.

Solution

A complex-valued function $f(x)$ on the interval $(-l, l)$ will have the following complex Fourier series representation.

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}$$

First, we will assume that $f(x)$ is real-valued and prove that $c_n = \overline{c_{-n}}$. Take the complex conjugate of both sides.

$$\overline{f(x)} = \overline{\sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}}$$

$f(x)$ is real, so $\overline{f(x)} = f(x)$.

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} \overline{c_n e^{in\pi x/l}} \\ &= \sum_{n=-\infty}^{\infty} \overline{c_n}\, \overline{e^{in\pi x/l}} \\ &= \sum_{n=-\infty}^{\infty} \overline{c_n}\, e^{-in\pi x/l} \end{aligned}$$

Substitute $n = -k$ in the series.

$$= \sum_{-k=-\infty}^{\infty} \overline{c_{-k}}\, e^{-i(-k)\pi x/l}$$

k essentially goes from $-\infty$ to ∞.

$$= \sum_{k=-\infty}^{\infty} \overline{c_{-k}}\, e^{ik\pi x/l}$$

Replacing the dummy variable k with n, we get

$$= \sum_{n=-\infty}^{\infty} \overline{c_{-n}}\, e^{in\pi x/l}.$$

We have shown that

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l} = \sum_{n=-\infty}^{\infty} \overline{c_{-n}}\, e^{in\pi x/l}.$$

Therefore,

$$c_n = \overline{c_{-n}}.$$

Secondly, we will assume that $c_n = \overline{c_{-n}}$ and prove that $f(x)$ is real-valued.

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l} = \sum_{n=-\infty}^{\infty} \overline{c_{-n}}\, e^{in\pi x/l}$$

Substitute $n = -k$ in the second series for $f(x)$.

$$f(x) = \sum_{-k=-\infty}^{\infty} \overline{c_{-(-k)}}\, e^{i(-k)\pi x/l}$$

k essentially goes from $-\infty$ and ∞.

$$\begin{aligned} &= \sum_{k=-\infty}^{\infty} \overline{c_k}\, e^{-ik\pi x/l} \\ &= \sum_{k=-\infty}^{\infty} \overline{c_k}\, \overline{e^{ik\pi x/l}} \\ &= \sum_{k=-\infty}^{\infty} \overline{c_k e^{ik\pi x/l}} \\ &= \overline{\sum_{k=-\infty}^{\infty} c_k e^{ik\pi x/l}} \end{aligned}$$

Replacing the dummy variable k with n, we get

$$\begin{aligned} &= \overline{\sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}} \\ &= \overline{f(x)}. \end{aligned}$$

Since $f(x) = \overline{f(x)}$, $f(x)$ is real-valued. Therefore, a complex-valued function $f(x)$ is real-valued if and only if its complex Fourier coefficients satisfy $c_n = \overline{c_{-n}}$.