Exercise 7

Show how the full Fourier series on $(-l, l)$ can be derived from the full series on $(-\pi, \pi)$ by changing variables $w = (\pi/l)x$. (This is called a *change of scale;* it means that one unit along the x axis becomes π/l units along the w axis.)

Solution

The full Fourier series on $(-\pi, \pi)$ for a function $\phi(w)$ is given by

$$\phi(w) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos nw + \sum_{n=1}^{\infty} B_n \sin nw,$$

where

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(w)\cos nw\, dw, \quad n = 0, 1, \ldots$$

and

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(w)\sin nw\, dw, \quad n = 1, 2, \ldots.$$

Make the prescribed substitution in the series and the integrals associated with it.

$$\begin{aligned} w &= \frac{\pi}{l}x & \quad\rightarrow\quad & \frac{l}{\pi}w = x \\ dw &= \frac{\pi}{l}\,dx & \quad\rightarrow\quad & \frac{l}{\pi}\,dw = dx \end{aligned}$$

As a result,

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l},$$

where

$$\begin{aligned} A_n &= \frac{1}{\pi}\int_{-l}^{l} \phi(x)\cos\frac{n\pi x}{l}\left(\frac{\pi}{l}\,dx\right), \quad n = 0, 1, \ldots \\ &= \frac{1}{l}\int_{-l}^{l} \phi(x)\cos\frac{n\pi x}{l}\,dx \end{aligned}$$

and

$$\begin{aligned} B_n &= \frac{1}{\pi}\int_{-l}^{l} \phi(x)\sin\frac{n\pi x}{l}\left(\frac{\pi}{l}\,dx\right), \quad n = 1, 2, \ldots \\ &= \frac{1}{l}\int_{-l}^{l} \phi(x)\sin\frac{n\pi x}{l}\,dx. \end{aligned}$$