Exercise 7

Show how the full Fourier series on $(-l, l)$ can be derived from the full series on $(-\pi, \pi)$ by changing variables $w = (\pi/l)x$. (This is called a change of scale; it means that one unit along the $x$ axis becomes $\pi/l$ units along the $w$ axis.)

Solution

The full Fourier series on $(-\pi, \pi)$ for a function $\phi(w)$ is given by

$$\phi(w) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos nw + \sum_{n=1}^{\infty} B_n \sin nw,$$

where

$$A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(w) \cos nw \, dw, \quad n = 0, 1, \ldots$$

and

$$B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(w) \sin nw \, dw, \quad n = 1, 2, \ldots.$$

Make the prescribed substitution in the series and the integrals associated with it.

$$w = \frac{\pi}{l}x \quad \rightarrow \quad \frac{l}{\pi}w = x$$

$$dw = \frac{\pi}{l} \, dx \quad \rightarrow \quad \frac{l}{\pi} \, dw = dx$$

As a result,

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l},$$

where

$$A_n = \frac{1}{\pi} \int_{-l}^{l} \phi(x) \cos \frac{n\pi x}{l} \left( \frac{\pi}{l} \, dx \right), \quad n = 0, 1, \ldots$$

$$= \frac{1}{l} \int_{-l}^{l} \phi(x) \cos \frac{n\pi x}{l} \, dx$$

and

$$B_n = \frac{1}{\pi} \int_{-l}^{l} \phi(x) \sin \frac{n\pi x}{l} \left( \frac{\pi}{l} \, dx \right), \quad n = 1, 2, \ldots$$

$$= \frac{1}{l} \int_{-l}^{l} \phi(x) \sin \frac{n\pi x}{l} \, dx.$$