Exercise 8

(a) Prove that differentiation switches even functions to odd ones, and odd functions to even ones.

(b) Prove the same for integration provided that we ignore the constant of integration.

Solution

Suppose $f(x)$ is an odd function so that $f(-x) = -f(x)$. Also, suppose $g(x)$ is an even function so that $g(-x) = g(x)$.

Part (a)

Consider the first derivative of f.

$$f'(x) = \frac{df(x)}{dx}$$

This is an even function because

$$f'(-x) = \frac{df(-x)}{d(-x)} = \frac{(-1)}{(-1)}\frac{df(x)}{dx} = f'(x).$$

Consider the first derivative of g.

$$g'(x) = \frac{dg(x)}{dx}$$

This is an odd function because

$$g'(-x) = \frac{dg(-x)}{d(-x)} = \frac{1}{(-1)}\frac{dg(x)}{dx} = -g'(x).$$

Therefore, differentiation switches even functions to odd ones, and odd functions to even ones.

Part (b)

Consider the integral of f.

$$F(x) = \int^x f(s)\,ds$$

This is an even function because

$$F(-x) = \int^{-x} f(s)\,ds = \int^x f(-r)\,(-dr) = \int^x [-f(r)]\,(-dr) = \int^x f(r)\,dr = F(x),$$

where the substitution, $s = -r$ and $ds = -dr$, was used. Now consider the integral of g.

$$G(x) = \int^x g(s)\,ds$$

This is an odd function because

$$G(-x) = \int^{-x} g(s)\,ds = \int^x g(-r)\,(-dr) = \int^x g(r)\,(-dr) = -\int^x g(r)\,dr = -G(x),$$

where the substitution, $s = -r$ and $ds = -dr$, was used. Therefore, integration switches even functions to odd ones, and odd functions to even ones as well.