Exercise 11

Find the full Fourier series of e^x on $(-l, l)$ in its real and complex forms. (*Hint:* It is convenient to find the complex form first.)

Solution

The Complex Fourier Series

The complex Fourier series of e^x is

$$e^x = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}.$$

To determine the coefficients c_n, multiply both sides by $e^{-im\pi x/l}$, where m is an integer. We assume that $n \neq m$ for now.

$$e^x e^{-im\pi x/l} = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l} e^{-im\pi x/l}$$

Integrate both sides with respect to x over the interval $(-l, l)$.

$$\int_{-l}^{l} e^x e^{-im\pi x/l}\, dx = \int_{-l}^{l} \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l} e^{-im\pi x/l}\, dx$$

Combine the exponential functions and bring the constants in front of the integral on the right side.

$$\int_{-l}^{l} e^{x(1-im\pi/l)}\, dx = \sum_{n=-\infty}^{\infty} c_n \int_{-l}^{l} e^{i(n-m)\pi x/l}\, dx$$

Evaluate the integrals.

$$\frac{1}{1-im\pi/l} e^{x(1-im\pi/l)}\Big|_{-l}^{l} = \sum_{n=-\infty}^{\infty} c_n \cdot \frac{1}{i(n-m)\pi/l} e^{i(n-m)\pi x/l}\Big|_{-l}^{l}$$

$$\frac{l}{l-im\pi}[e^{l(1-im\pi/l)} - e^{-l(1-im\pi/l)}] = \sum_{n=-\infty}^{\infty} c_n \cdot \frac{l}{i(n-m)\pi}[e^{i(n-m)\pi} - e^{-i(n-m)\pi}]$$

$$\frac{l(l+im\pi)}{(l-im\pi)(l+im\pi)}(e^{l-im\pi} - e^{-l+im\pi}) = \sum_{n=-\infty}^{\infty} c_n \cdot \frac{l}{i(n-m)\pi}\{\cancel{\cos[(n-m)\pi]} + \underbrace{i\sin[(n-m)\pi]}_{=0} - \cancel{\cos[(n-m)\pi]} + \underbrace{i\sin[(n-m)\pi]}_{=0}\}$$

We find that every term in the infinite series is zero if $n \neq m$ as a result of the integral. The $n = m$ term is all that remains on the right side.

$$\frac{l(l+in\pi)}{(l-in\pi)(l+in\pi)}(e^{l-in\pi} - e^{-l+in\pi}) = c_n \int_{-l}^{l} e^0\, dx$$

Evaluate the integral on the right side.

$$\frac{l(l+in\pi)}{l^2+n^2\pi^2}[e^{i(-il-n\pi)}-e^{-i(-il-n\pi)}]=c_n\cdot 2l$$

Divide both sides by $2l$ to solve for c_n.

$$\begin{aligned}
c_n &= \frac{l+in\pi}{l^2+n^2\pi^2}\cdot i\frac{e^{i(-il-n\pi)}-e^{-i(-il-n\pi)}}{2i} \\
&= \frac{l+in\pi}{l^2+n^2\pi^2}\cdot i\sin(-il-n\pi) \\
&= \frac{l+in\pi}{l^2+n^2\pi^2}\cdot i(-1)\sin(il+n\pi) \\
&= \frac{l+in\pi}{l^2+n^2\pi^2}\cdot i(-1)[\sin il\underbrace{\cos n\pi}_{=(-1)^n}+\cos il\underbrace{\sin n\pi}_{=0}] \\
&= \frac{l+in\pi}{l^2+n^2\pi^2}\cdot i(-1)(-1)^n\sin il
\end{aligned}$$

Make use of the identity $\sin il = i\sinh l$.

$$\begin{aligned}
&= \frac{l+in\pi}{l^2+n^2\pi^2}\cdot i(-1)(-1)^n i\sinh l \\
&= (-1)^n\frac{l+in\pi}{l^2+n^2\pi^2}\sinh l
\end{aligned}$$

Therefore, the complex Fourier series of e^x on $(-l,l)$ is

$$e^x=\sum_{n=-\infty}^{\infty}(-1)^n\frac{l+in\pi}{l^2+n^2\pi^2}(\sinh l)e^{in\pi x/l}.$$

The Real Fourier Series

To obtain the real Fourier series, split up the sum.

$$e^x=\sum_{n=-\infty}^{-1}(-1)^n\frac{l+in\pi}{l^2+n^2\pi^2}(\sinh l)e^{in\pi x/l}+\frac{\sinh l}{l}+\sum_{n=1}^{\infty}(-1)^n\frac{l+in\pi}{l^2+n^2\pi^2}(\sinh l)e^{in\pi x/l}$$

Substitute $n=-k$ in the first sum and $n=k$ in the last sum.

$$=\sum_{-k=-\infty}^{-1}(-1)^{-k}\frac{l+i(-k)\pi}{l^2+(-k)^2\pi^2}(\sinh l)e^{i(-k)\pi x/l}+\frac{\sinh l}{l}+\sum_{k=1}^{\infty}(-1)^k\frac{l+ik\pi}{l^2+k^2\pi^2}(\sinh l)e^{ik\pi x/l}$$

k essentially runs from 1 to ∞ in the first sum. Also, $(-1)^{-k}=(-1)^k$.

$$=\sum_{k=1}^{\infty}(-1)^k\frac{l-ik\pi}{l^2+k^2\pi^2}(\sinh l)e^{-ik\pi x/l}+\frac{\sinh l}{l}+\sum_{k=1}^{\infty}(-1)^k\frac{l+ik\pi}{l^2+k^2\pi^2}(\sinh l)e^{ik\pi x/l}$$

Combine the two sums.

$$=\frac{\sinh l}{l}+\sum_{k=1}^{\infty}\left[(-1)^k\frac{l-ik\pi}{l^2+k^2\pi^2}(\sinh l)e^{-ik\pi x/l}+(-1)^k\frac{l+ik\pi}{l^2+k^2\pi^2}(\sinh l)e^{ik\pi x/l}\right]$$

Factor the summand.

$$e^x = \frac{\sinh l}{l} + \sum_{k=1}^{\infty}(-1)^k \frac{\sinh l}{l^2+k^2\pi^2}\left[(l-ik\pi)e^{-ik\pi x/l} + (l+ik\pi)e^{ik\pi x/l}\right]$$

Use Euler's formula to write the exponential functions in terms of sine and cosine.

$$= \frac{\sinh l}{l} + \sum_{k=1}^{\infty}(-1)^k \frac{\sinh l}{l^2+k^2\pi^2}\left[(l-ik\pi)\left(\cos\frac{k\pi x}{l} - i\sin\frac{k\pi x}{l}\right) + (l+ik\pi)\left(\cos\frac{k\pi x}{l} + i\sin\frac{k\pi x}{l}\right)\right]$$

Expand the terms in square brackets—the cross-terms cancel out.

$$= \frac{\sinh l}{l} + \sum_{k=1}^{\infty}(-1)^k \frac{\sinh l}{l^2+k^2\pi^2}\left(2l\cos\frac{k\pi x}{l} - 2k\pi\sin\frac{k\pi x}{l}\right)$$

Factor 2 from the parentheses and bring the constants in front of the sum.

$$= \frac{\sinh l}{l} + (2\sinh l)\sum_{k=1}^{\infty}\frac{(-1)^k}{l^2+k^2\pi^2}\left(l\cos\frac{k\pi x}{l} - k\pi\sin\frac{k\pi x}{l}\right)$$

Replacing the dummy index k with n, therefore, the real Fourier series for e^x on $(-l, l)$ is

$$e^x = \frac{\sinh l}{l} + (2\sinh l)\sum_{n=1}^{\infty}\frac{(-1)^n}{l^2+n^2\pi^2}\left(l\cos\frac{n\pi x}{l} - n\pi\sin\frac{n\pi x}{l}\right).$$

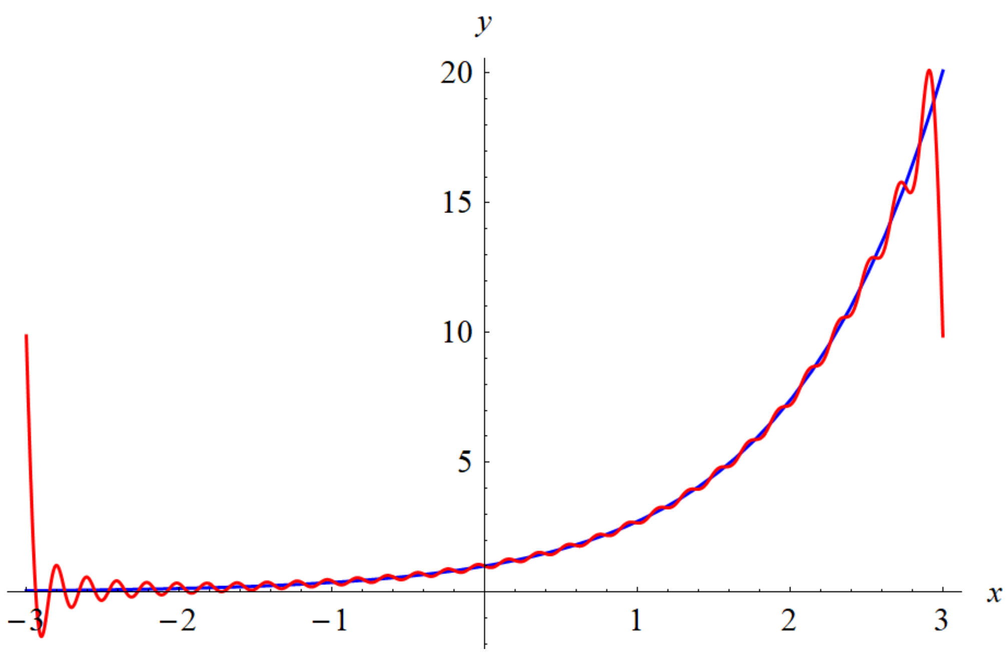

Figure 1: This is a sample plot of $y = e^x$ on $(-3, 3)$ in blue. An approximation to the Fourier series of e^x is plotted in red, where only the first 30 terms in the infinite series have been used.