Exercise 13

Repeat Exercise 11 for \( \sin x \). Assume that \( l \) is not an integer multiple of \( \pi \). \((Hint: \ First\ find\ the\ series\ for\ e^{ix}.)\)

Solution

The Complex Fourier Series

The complex Fourier series of \( \sin x \) is

\[
\sin x = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}.
\]

To determine the coefficients \( c_n \), multiply both sides by \( e^{-im\pi x/l} \), where \( m \) is an integer. We assume that \( n \neq m \) for now.

\[
(sin x)e^{-im\pi x/l} = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l} e^{-im\pi x/l}
\]

Integrate both sides with respect to \( x \) over the interval \((-l,l)\).

\[
\int_{-l}^{l} (sin x)e^{-im\pi x/l} \, dx = \int_{-l}^{l} \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l} e^{-im\pi x/l} \, dx
\]

Write the sine function in terms of exponential functions.

\[
\int_{-l}^{l} \frac{e^{ix} - e^{-ix}}{2i} e^{-im\pi x/l} \, dx = \int_{-l}^{l} \sum_{n=-\infty}^{\infty} c_n e^{i(n-m)\pi x/l} \, dx
\]

Bring the constants in front of the integrals.

\[
\frac{1}{2i} \int_{-l}^{l} (e^{ix} - e^{-ix})e^{-im\pi x/l} \, dx = \sum_{n=-\infty}^{\infty} c_n \int_{-l}^{l} e^{i(n-m)\pi x/l} \, dx
\]

Multiply both sides by \( 2i \).

\[
\int_{-l}^{l} (e^{ix} e^{-im\pi x/l} - e^{-ix} e^{-im\pi x/l}) \, dx = 2i \sum_{n=-\infty}^{\infty} c_n \int_{-l}^{l} e^{i(n-m)\pi x/l} \, dx
\]

Split up the integral on the left side into two and combine the exponential functions.

\[
\int_{-l}^{l} e^{ix(1-m\pi/l)} \, dx - \int_{-l}^{l} e^{-ix(1+m\pi/l)} \, dx = 2i \sum_{n=-\infty}^{\infty} c_n \int_{-l}^{l} e^{i(n-m)\pi x/l} \, dx
\]

Evaluate the integrals.

\[
\left. \frac{1}{i(1-m\pi/l)} e^{ix(1-m\pi/l)} \right|_{-l}^{l} - \left. \frac{1}{-i(1+m\pi/l)} e^{-ix(1+m\pi/l)} \right|_{-l}^{l} = 2i \sum_{n=-\infty}^{\infty} c_n \cdot \frac{1}{i(n-m)\pi/l} e^{i(n-m)\pi x/l} \left|_{-l}^{l} \right.
\]

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Multiply both sides by \( i/l \).

\[
\frac{1}{l-m\pi}e^{ix(l-m\pi/i)} \bigg|_{-l}^{l} + \frac{1}{l+m\pi}e^{-ix(l+m\pi/i)} \bigg|_{-l}^{l} = -\frac{2}{l} \sum_{n=-\infty}^{\infty} c_n \cdot \frac{1}{i(n-m)\pi/l} e^{i(n-m)\pi x/l} \bigg|_{-l}^{l}
\]

Multiply both sides by the least common denominator \((l-m\pi)(l+m\pi) = l^2 - m^2\pi^2\).

\[
(l+m\pi)e^{ix(l-m\pi/i)} \bigg|_{-l}^{l} + (l-m\pi)e^{-ix(l+m\pi/i)} \bigg|_{-l}^{l} = -\frac{2}{l}(l^2 - m^2\pi^2) \sum_{n=-\infty}^{\infty} c_n \cdot \frac{1}{i(n-m)\pi/l} e^{i(n-m)\pi x/l} \bigg|_{-l}^{l}
\]

Now plug in the limits.

\[
(l+m\pi)[e^{i(l-m\pi)} - e^{-i(l-m\pi)}] + (l-m\pi)[e^{-i(l+m\pi)} - e^{i(l+m\pi)}] = -\frac{2}{l}(l^2 - m^2\pi^2) \sum_{n=-\infty}^{\infty} c_n \cdot \frac{1}{i(n-m)\pi/l} [e^{i(n-m)\pi} - e^{-i(n-m)\pi}]
\]

Use Euler’s formula to write each exponential function in terms of sine and cosine.

\[
(l + m\pi) \left\{ [\cos(l-m\pi) + i \sin(l-m\pi)] - [\cos(l-m\pi) - i \sin(l-m\pi)] \right\}
+ (l - m\pi) \left\{ [\cos(l + m\pi) - i \sin(l + m\pi)] - [\cos(l + m\pi) + i \sin(l + m\pi)] \right\}
= -\frac{2}{l}(l^2 - m^2\pi^2) \sum_{n=-\infty}^{\infty} c_n \cdot \frac{1}{i(n-m)\pi/l} \left\{ \cos[(n-m)\pi] + i \underbrace{\sin[(n-m)\pi]}_{=0} \right\}
- \cos[(n-m)\pi] + i \underbrace{\sin[(n-m)\pi]}_{=0}
\]

We find that every term in the infinite series is zero if \( n \neq m \) as a result of the integral. The \( n = m \) term is all that remains on the right side.

\[
(l + n\pi) \left\{ [\cos(l-n\pi) + i \sin(l-n\pi)] - [\cos(l-n\pi) - i \sin(l-n\pi)] \right\}
+ (l - n\pi) \left\{ [\cos(l + n\pi) - i \sin(l + n\pi)] - [\cos(l + n\pi) + i \sin(l + n\pi)] \right\}
= -\frac{2}{l}(l^2 - n^2\pi^2)c_n \int_{-l}^{l} e^0 \, dx
\]

Simplify the left side and evaluate the integral on the right side.

\[
(l + n\pi)[2i \sin(l-n\pi)] + (l - n\pi)[-2i \sin(l + n\pi)] = -\frac{2}{l}(l^2 - n^2\pi^2)c_n \cdot 2l
\]

Divide both sides by \( 2i \).

\[
(l + n\pi) \sin(l-n\pi) - (l - n\pi) \sin(l + n\pi) = -\frac{2}{l}(l^2 - n^2\pi^2)c_n
\]

Expand the sines.

\[
(l + n\pi)(\sin l \cos n\pi - \cos l \sin n\pi) - (l - n\pi)(\sin l \cos n\pi + \cos l \sin n\pi) = -\frac{2}{l}(l^2 - n^2\pi^2)c_n
\]

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Simplify the left side.

\[(l + n\pi)(-1)^n \sin l - (l - n\pi)(-1)^n \sin l = -\frac{2}{l}(l^2 - n^2 \pi^2)c_n\]

Combine like-terms.

\[2n\pi(-1)^n \sin l = -\frac{2}{l}(l^2 - n^2 \pi^2)c_n\]

Solving for \(c_n\), we have

\[c_n = \frac{i\pi(-1)^{n+1} \sin l}{l^2 - n^2 \pi^2}.\]

Therefore, the complex Fourier series of \(\sin x\) on \((-l, l)\) is

\[\sin x = \sum_{n=-\infty}^{\infty} \frac{i\pi(-1)^{n+1} \sin l}{l^2 - n^2 \pi^2} e^{i\pi x/l}.\]

**The Real Fourier Series**

To obtain the real Fourier series, split up the sum.

\[\sin x = \sum_{n=-\infty}^{-1} \frac{i\pi(-1)^{n+1} \sin l}{l^2 - n^2 \pi^2} e^{i\pi x/l} + \sum_{n=1}^{\infty} \frac{i\pi(-1)^{n+1} \sin l}{l^2 - n^2 \pi^2} e^{i\pi x/l}\]

Substitute \(n = -k\) in the first sum and \(n = k\) in the last sum.

\[= \sum_{-k=-\infty}^{-1} \frac{-i(-k)\pi(-1)^{-k+1} \sin l}{l^2 - (-k)^2 \pi^2} e^{(-k)\pi x/l} + \sum_{k=1}^{\infty} \frac{i(k\pi(-1)^{k+1} \sin l}{l^2 - k^2 \pi^2} e^{ik\pi x/l}\]

\(k\) essentially runs from 1 to \(\infty\) in the first sum. Also, \((-1)^{-k+1} = (-1)^{k+1}\).

\[= -\sum_{k=1}^{\infty} \frac{i(k\pi(-1)^{k+1} \sin l}{l^2 - k^2 \pi^2} e^{-ik\pi x/l} + \sum_{k=1}^{\infty} \frac{i(k\pi(-1)^{k+1} \sin l}{l^2 - k^2 \pi^2} e^{ik\pi x/l}\]

Combine the two sums.

\[= \sum_{k=1}^{\infty} \left[ -\frac{i\pi(-1)^{k+1} \sin l}{l^2 - k^2 \pi^2} e^{-ik\pi x/l} + \frac{i(k\pi(-1)^{k+1} \sin l}{l^2 - k^2 \pi^2} e^{ik\pi x/l}\right]\]

Factor the summand.

\[= \sum_{k=1}^{\infty} \frac{i(k\pi(-1)^{k+1} \sin l}{l^2 - k^2 \pi^2} (-e^{-ik\pi x/l} + e^{ik\pi x/l})\]

Use Euler’s formula to write the exponential functions in terms of sine and cosine.

\[= \sum_{k=1}^{\infty} \frac{i(k\pi(-1)^{k+1} \sin l}{l^2 - k^2 \pi^2} \left[ -\left(\cos \frac{k\pi x}{l} - i \sin \frac{k\pi x}{l}\right) + \left(\cos \frac{k\pi x}{l} + i \sin \frac{k\pi x}{l}\right)\right]\]
Simplify the result.

\[
\sin x = \sum_{k=1}^{\infty} \frac{ik\pi(-1)(-1)^k \sin l}{l^2 - k^2\pi^2} \left( 2i \sin \frac{k\pi x}{l} \right)
\]

\[
= \sum_{k=1}^{\infty} \frac{2k\pi(-1)^k \sin l}{l^2 - k^2\pi^2} \sin \frac{k\pi x}{l}
\]

\[
= (2\pi \sin l) \sum_{k=1}^{\infty} \frac{k(-1)^k}{l^2 - k^2\pi^2} \sin \frac{k\pi x}{l}
\]

Replacing the dummy index \( k \) with \( n \), therefore, the real Fourier series for \( \sin x \) on \((-l,l)\) is

\[
\sin x = (2\pi \sin l) \sum_{n=1}^{\infty} \frac{n(-1)^n}{l^2 - n^2\pi^2} \sin \frac{n\pi x}{l}.
\]

Figure 1: This is a sample plot of \( y = \sin x \) on \((-3,3)\) in blue. An approximation to the Fourier series of \( \sin x \) is plotted in red, where only the first 3 terms in the infinite series have been used.