Exercise 15

Use the same idea as in Exercises 12 and 13 to show that none of the eigenvalues of the fourth-order operator $+d^4/dx^4$ with the boundary conditions $X(0) = X(l) = X''(0) = X''(l) = 0$ are negative.

Solution

The eigenvalue problem for d^4/dx^4 is

$$\frac{d^4}{dx^4}X = \lambda X. \tag{1}$$

Consider the integral of $X''''(x)X(x)$ with respect to x over the interval $(0, l)$ and use integration by parts.

$$\begin{aligned}\int_0^l X''''X\,dx &= X'''X\Big|_0^l - \int_0^l X'''X'\,dx \\ &= X'''(l)X(l) - X'''(0)X(0) - \int_0^l X'''X'\,dx\end{aligned}$$

$X(l) = 0$ and $X(0) = 0$, so the first two terms vanish.

$$= -\int_0^l X'''X'\,dx$$

Use integration by parts once more.

$$\begin{aligned}&= -\left[X''X'\Big|_0^l - \int_0^l X''^2\,dx\right] \\ &= -\left[X''(l)X'(l) - X''(0)X'(0) - \int_0^l X''^2\,dx\right]\end{aligned}$$

$X''(l) = 0$ and $X''(0) = 0$, so the first two terms vanish.

$$= \int_0^l X''^2\,dx$$

Substitute equation (1) into the left side.

$$\begin{aligned}\int_0^l (\lambda X)X\,dx &= \int_0^l X''^2\,dx \\ \int_0^l \lambda X^2\,dx &= \int_0^l X''^2\,dx\end{aligned}$$

λ has to be positive because X^2 and X''^2 are positive. Therefore, none of the eigenvalues are negative.