Exercise 3

Consider $u_{tt} = c^2 u_{xx}$ for $0 < x < l$, with the boundary conditions $u(0,t) = 0$, $u_x(l,t) = 0$ and the initial conditions $u(x,0) = x$, $u_t(x,0) = 0$. Find the solution explicitly in series form.

Solution

The PDE and its boundary conditions are linear and homogeneous, so the method of separation of variables can be applied. Assume a product solution of the form $u(x,t) = X(x)T(t)$ and plug it into the PDE

$$u_{tt} = c^2 u_{xx} \quad \rightarrow \quad XT'' = c^2 X'' T$$

and the boundary conditions.

$$\begin{array}{ccccc} u(0,t) = 0 & \rightarrow & X(0)T(t) = 0 & \rightarrow & X(0) = 0 \\ u_x(l,t) = 0 & \rightarrow & X'(l)T(t) = 0 & \rightarrow & X'(l) = 0 \end{array}$$

Now separate variables in the PDE: bring all functions of t and constants to the left side and all functions of x to the right side. The final answer would be the same if c^2 were brought to the right side.

$$\frac{T''}{c^2 T} = \frac{X''}{X}$$

The only way a function of t on the left can be equal to a function of x on the right is if both sides are equal to a constant λ.

$$\frac{T''}{c^2 T} = \frac{X''}{X} = \lambda$$

Values of λ for which $X(0) = 0$ and $X'(l) = 0$ are satisfied are called the eigenvalues, and the nontrivial functions $X(x)$ associated with them are called the eigenfunctions.

Determination of Positive Eigenvalues: $\lambda = \mu^2$

Assuming λ is positive, the differential equation for X becomes

$$\frac{X''}{X} = \mu^2.$$

Multiply both sides by X.

$$X'' = \mu^2 X$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \mu x + C_2 \sinh \mu x$$

Now use the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} X(0) &= C_1 = 0 \\ X'(l) &= \mu(C_1 \sinh \mu l + C_2 \cosh \mu l) = 0 \end{aligned}$$

The second equation simplifies to $C_2 \mu \cosh \mu l = 0$, which can only be satisfied if $C_2 = 0$. Only the trivial solution $X(x) = 0$ results from considering positive values for λ, so there are no positive eigenvalues.

Determination of the Zero Eigenvalue: $\lambda = 0$

Assuming λ is zero, the differential equation for X becomes

$$\frac{X''}{X} = 0.$$

Multiply both sides by X.

$$X'' = 0$$

The general solution is a linear function.

$$X(x) = C_3 x + C_4$$

Now use the boundary conditions to determine C_3 and C_4.

$$\begin{aligned} X(0) &= C_4 = 0 \\ X'(l) &= C_3 = 0 \end{aligned}$$

Only the trivial solution $X(x) = 0$ is obtained, so zero is not an eigenvalue.

Determination of Negative Eigenvalues: $\lambda = -\gamma^2$

Assuming λ is negative, the differential equation for X becomes

$$\frac{X''}{X} = -\gamma^2.$$

Multiply both sides by X.

$$X'' = -\gamma^2 X$$

The general solution can be written in terms of sine and cosine.

$$X(x) = C_5 \cos \gamma x + C_6 \sin \gamma x$$

Apply the boundary conditions here to determine C_5 and C_6.

$$\begin{aligned} X(0) &= C_5 = 0 \\ X'(l) &= \gamma(-C_5 \sin \gamma l + C_6 \cos \gamma l) = 0 \end{aligned}$$

The second equation reduces to

$$C_6 \gamma \cos \gamma l = 0.$$

In order to avoid getting the trivial solution, we insist that $C_6 \neq 0$. The equation for γ is then

$$\begin{aligned} \gamma \cos \gamma l &= 0 \\ \gamma l &= \frac{1}{2}(2n-1)\pi \quad \rightarrow \quad \gamma_n = \frac{\pi}{2l}(2n-1), \quad n = 1, 2, \ldots. \end{aligned}$$

The eigenfunctions associated with these eigenvalues are

$$X(x) = C_6 \sin \gamma x \quad \rightarrow \quad X_n(x) = \sin\left[\frac{\pi}{2l}(2n-1)x\right], \quad n = 1, 2, \ldots.$$

The ODE for $T(t)$ will now be solved.

$$\frac{T''}{c^2T} = -\gamma^2$$

Multiply both sides by c^2T.

$$T'' = -c^2\gamma^2 T$$

The general solution can be written in terms of sine and cosine.

$$T(t) = C_7 \cos c\gamma t + C_8 \sin c\gamma t \quad \rightarrow \quad T_n(t) = C_7 \cos\left[\frac{\pi}{2l}(2n-1)ct\right] + C_8 \sin\left[\frac{\pi}{2l}(2n-1)ct\right]$$

According to the principle of linear superposition, the solution to the PDE for $u(x,t)$ is a linear combination of all products $T_n(t)X_n(x)$ over all the eigenvalues.

$$u(x,t) = \sum_{n=1}^{\infty}\left\{A_n \cos\left[\frac{\pi}{2l}(2n-1)ct\right] + B_n \sin\left[\frac{\pi}{2l}(2n-1)ct\right]\right\}\sin\left[\frac{\pi}{2l}(2n-1)x\right]$$

Now we will use the initial conditions to determine the coefficients, A_n and B_n. $u_t(x,0) = 0$ will be used first since it is homogeneous. Take a derivative of the solution with respect to t.

$$u_t(x,t) = \sum_{n=1}^{\infty}\left\{-\frac{\pi}{2l}(2n-1)cA_n \sin\left[\frac{\pi}{2l}(2n-1)ct\right] + \frac{\pi}{2l}(2n-1)cB_n \cos\left[\frac{\pi}{2l}(2n-1)ct\right]\right\}\sin\left[\frac{\pi}{2l}(2n-1)x\right]$$

Setting $t = 0$ and using the initial condition,

$$u_t(x,0) = \sum_{n=1}^{\infty}\frac{\pi}{2l}(2n-1)cB_n \sin\left[\frac{\pi}{2l}(2n-1)x\right] = 0,$$

we find that $B_n = 0$. As a result, the general solution reduces to

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos\left[\frac{\pi}{2l}(2n-1)ct\right]\sin\left[\frac{\pi}{2l}(2n-1)x\right].$$

Set $t = 0$ and use the other initial condition now.

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left[\frac{\pi}{2l}(2n-1)x\right] = x$$

To determine A_n, multiply both sides by $\sin\gamma_m x$, where m is an integer. For now we assume that $n \neq m$.

$$\sum_{n=1}^{\infty} A_n \sin\left[\frac{\pi}{2l}(2n-1)x\right]\sin\left[\frac{\pi}{2l}(2m-1)x\right] = x\sin\left[\frac{\pi}{2l}(2m-1)x\right]$$

Integrate both sides with respect to x over the domain the PDE is defined.

$$\int_0^l \sum_{n=1}^{\infty} A_n \sin\left[\frac{\pi}{2l}(2n-1)x\right]\sin\left[\frac{\pi}{2l}(2m-1)x\right]dx = \int_0^l x\sin\left[\frac{\pi}{2l}(2m-1)x\right]dx$$

Bring the constants in front of the integral on the left side and use integration by parts on the right side.

$$\begin{aligned} v &= x & dw &= \sin\left[\frac{\pi}{2l}(2m-1)x\right]dx \\ dv &= dx & w &= -\frac{2l}{\pi(2m-1)}\cos\left[\frac{\pi}{2l}(2m-1)x\right] \end{aligned}$$

Use the formula $\int v\,dw = vw - \int w\,dv$.

$$\sum_{n=1}^{\infty} A_n \int_0^l \sin\left[\frac{\pi}{2l}(2n-1)x\right]\sin\left[\frac{\pi}{2l}(2m-1)x\right]dx$$
$$= \underbrace{-\frac{2l}{\pi(2m-1)}x\cos\left[\frac{\pi}{2l}(2m-1)x\right]\Bigg|_0^l}_{=0} - \int_0^l \left\{-\frac{2l}{\pi(2m-1)}\cos\left[\frac{\pi}{2l}(2m-1)x\right]\right\}dx$$

Use the product-to-sum formula,

$$\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)],$$

on the left side.

$$\sum_{n=1}^{\infty} A_n \int_0^l \frac{1}{2}\left\{\cos\left[\frac{\pi}{2l}(2n-1)x - \frac{\pi}{2l}(2m-1)x\right] - \cos\left[\frac{\pi}{2l}(2n-1)x + \frac{\pi}{2l}(2m-1)x\right]\right\}dx$$
$$= \frac{2l}{\pi(2m-1)}\int_0^l \cos\left[\frac{\pi}{2l}(2m-1)x\right]dx$$

Bring the factor of 1/2 in front of the integral, simplify the cosines' arguments, and split up the integral into two on the left side. Evaluate the integral on the right side.

$$\sum_{n=1}^{\infty} \frac{A_n}{2}\left[\int_0^l \cos\frac{(n-m)\pi x}{l}\,dx - \int_0^l \cos\frac{(n+m-1)\pi x}{l}\,dx\right]$$
$$= \frac{2l}{\pi(2m-1)}\cdot\frac{2l}{\pi(2m-1)}\sin\left[\frac{\pi}{2l}(2m-1)x\right]\Bigg|_0^l$$

Evaluate the two integrals on the left side.

$$\sum_{n=1}^{\infty} \frac{A_n}{2}\Bigg[\underbrace{\frac{l}{(n-m)\pi}\sin\frac{(n-m)\pi x}{l}\Bigg|_0^l}_{=0} - \underbrace{\frac{l}{(n+m-1)\pi}\sin\frac{(n+m-1)\pi x}{l}\Bigg|_0^l}_{=0}\Bigg]$$
$$= \frac{4l^2}{\pi^2(2m-1)^2}\sin\left[\frac{\pi}{2}(2m-1)\right]$$

We find that every term in the infinite series is zero if $n \neq m$ as a result of the integral. The $n = m$ term is all that remains on the left side.

$$A_n \int_0^l \sin^2\left[\frac{\pi}{2l}(2n-1)x\right]dx = \frac{4l^2}{\pi^2(2n-1)^2}\sin\left[\frac{\pi}{2}(2n-1)\right]$$

Write the integrand in terms of cosine with the double angle formula $\sin^2 a = 1 - \cos 2a$. The sine on the right side can be written as $(-1)^{n+1}$.

$$A_n \int_0^l \frac{1}{2}\left\{1 - \cos\left[\frac{\pi}{l}(2n-1)x\right]\right\}dx = \frac{4l^2}{\pi^2(2n-1)^2}(-1)^{n+1}$$

Multiply both sides by 2 and evaluate the last integral.

$$A_n \left\{ x - \frac{l}{\pi(2n-1)} \sin\left[\frac{\pi}{l}(2n-1)x\right] \right\}\Bigg|_0^l = \frac{8l^2}{\pi^2(2n-1)^2}(-1)^{n+1}$$

$$A_n \cdot l = \frac{8l^2}{\pi^2(2n-1)^2}(-1)^{n+1}$$

Divide both sides by l to solve for A_n.

$$A_n = \frac{8l}{\pi^2(2n-1)^2}(-1)^{n+1}$$

Now that A_n and B_n have been determined, the solution to the PDE is known.

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} A_n \cos\left[\frac{\pi}{2l}(2n-1)ct\right] \sin\left[\frac{\pi}{2l}(2n-1)x\right] \\ &= \sum_{n=1}^{\infty} \frac{8l}{\pi^2(2n-1)^2}(-1)^{n+1} \cos\left[\frac{\pi}{2l}(2n-1)ct\right] \sin\left[\frac{\pi}{2l}(2n-1)x\right] \end{aligned}$$

Therefore,

$$u(x,t) = \frac{8l}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \cos\left[\frac{\pi}{2l}(2n-1)ct\right] \sin\left[\frac{\pi}{2l}(2n-1)x\right].$$