Exercise 5

(a) Show that the boundary conditions $u(0,t) = 0$, $u_x(l,t) = 0$ lead to the eigenfunctions $(\sin(\pi x/2l), \sin(3\pi x/2l), \sin(5\pi x/2l), \dots)$.

(b) If $\phi(x)$ is any function on $(0, l)$, derive the expansion

$$\phi(x) = \sum_{n=0}^{\infty} C_n \sin\left\{\left(n + \frac{1}{2}\right)\frac{\pi x}{l}\right\} \quad (0 < x < l)$$

by the following method. Extend $\phi(x)$ to the function $\tilde{\phi}$ defined by $\tilde{\phi}(x) = \phi(x)$ for $0 \le x \le l$ and $\tilde{\phi}(x) = \phi(2l - x)$ for $l \le x \le 2l$. (This means that you are extending it *evenly across* $x = l$.) Write the Fourier sine series for $\tilde{\phi}(x)$ on the interval $(0, 2l)$ and write the formula for the coefficients.

(c) Show that every second coefficient vanishes.

(d) Rewrite the formula for C_n as an integral of the original function $\phi(x)$ on the interval $(0, l)$.