Exercise 5

(a) Show that the boundary conditions \( u(0,t) = 0, \) \( u_x(l,t) = 0 \) lead to the eigenfunctions 
\( \sin(\pi x/2l), \sin(3\pi x/2l), \sin(5\pi x/2l), \ldots \).

(b) If \( \phi(x) \) is any function on \((0,l)\), derive the expansion
\[
\phi(x) = \sum_{n=0}^{\infty} C_n \sin \left\{ \left( n + \frac{1}{2} \right) \frac{\pi x}{l} \right\} \quad (0 < x < l)
\]
by the following method. Extend \( \phi(x) \) to the function \( \tilde{\phi}(x) = \phi(x) \) for 
\( 0 \leq x \leq l \) and \( \tilde{\phi}(x) = \phi(2l - x) \) for \( l \leq x \leq 2l \). (This means that you are extending it evenly 
across \( x = l \).) Write the Fourier sine series for \( \tilde{\phi}(x) \) on the interval \((0,2l)\) and write the 
formula for the coefficients.

(c) Show that every second coefficient vanishes.

(d) Rewrite the formula for \( C_n \) as an integral of the original function \( \phi(x) \) on the interval \((0,l)\).

Solution

Part (a)

Applying the method of separation of variables to either the wave or diffusion equation results in 
the eigenvalue problem \( X'' = \lambda X \) with the boundary conditions, \( X(0) = 0 \) and \( X'(l) = 0 \). 
Determine if there are any positive eigenvalues by setting \( \lambda = \mu^2 \) and solving the resulting 
boundary value problem.

\[
X'' = \mu^2 X \\
X(x) = C_1 \cosh \mu x + C_2 \sinh \mu x
\]

Use the boundary conditions to determine \( C_1 \) and \( C_2 \).

\[
X(0) = C_1 = 0 \\
X'(l) = \mu (C_1 \sinh \mu l + C_2 \cosh \mu l) = 0
\]

The second equation reduces to \( C_2 \cosh \mu l = 0 \). Since hyperbolic cosine is not oscillatory, the only 
way this equation is satisfied is if \( C_2 = 0 \). The trivial solution is obtained, so there are no positive 
eigenvalues. Set \( \lambda = 0 \) to determine if zero is an eigenvalue.

\[
X'' = 0 \\
X'(x) = C_3.
\]

Apply the boundary condition \( x'(l) = 0 \) to determine \( C_3 \).

\[
X'(l) = C_3 = 0
\]

Integrate both sides of the previous equation with respect to \( x \) once more.

\[
X(x) = C_4
\]
Apply the boundary condition \( x(0) = 0 \) to determine \( C_4 \).

\[
X(0) = C_4 = 0
\]

The trivial solution \( X(x) = 0 \) is obtained again, so zero is not an eigenvalue. Determine if there are any negative eigenvalues by setting \( \lambda = -\gamma^2 \).

\[
X'' = -\gamma^2 X
\]

\[
X(x) = C_5 \cos \gamma x + C_6 \sin \gamma x
\]

Use the boundary conditions to determine \( C_5 \) and \( C_6 \).

\[
X(0) = C_5 = 0
\]

\[
X'(l) = \gamma (-C_5 \sin \gamma l + C_6 \cos \gamma l) = 0
\]

The second equation reduces to

\[
C_6 \gamma \cos \gamma l = 0
\]

\[
\cos \gamma l = 0
\]

\[
\gamma l = \frac{\pi}{2} (2n + 1), \quad n = 0, 1, 2, \ldots
\]

\[
\gamma_n = \frac{\pi}{2l} (2n + 1), \quad n = 0, 1, 2, \ldots
\]

The eigenfunctions associated with the eigenvalues \( \lambda = -\gamma^2 \) are

\[
X(x) = C_6 \sin \mu x \quad \rightarrow \quad X_n(x) = \sin \left[ \frac{\pi}{2l} (2n + 1) x \right], \quad n = 0, 1, 2, \ldots
\]

That is,

\[
X_0(x) = \sin \frac{\pi x}{2l}
\]

\[
X_1(x) = \sin \frac{3\pi x}{2l}
\]

\[
X_2(x) = \sin \frac{5\pi x}{2l}
\]

\[\vdots\]

**Part (b)**

Here we introduce \( \tilde{\phi}(x) \), which is defined as

\[
\tilde{\phi}(x) = \begin{cases} 
\phi(x) & \text{for } 0 \leq x \leq l \\
\phi(2l - x) & \text{for } l \leq x \leq 2l
\end{cases}
\]

The Fourier sine series for this function over the interval \((0, 2l)\) is

\[
\tilde{\phi}(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{2l}.
\]
To obtain the coefficients $B_n$, multiply both sides by $\sin(m\pi x/2l)$, where $m$ is an integer,

$$\tilde{\phi}(x) \sin \frac{m\pi x}{2l} = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{2l} \sin \frac{m\pi x}{2l}$$

and then integrate both sides with respect to $x$ from 0 to $2l$.

$$\int_0^{2l} \tilde{\phi}(x) \sin \frac{m\pi x}{2l} \, dx = \int_0^{2l} \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{2l} \sin \frac{m\pi x}{2l} \, dx$$

$$= \sum_{n=1}^{\infty} B_n \int_0^{2l} \sin \frac{n\pi x}{2l} \sin \frac{m\pi x}{2l} \, dx$$

Because the sine functions are orthogonal, the integral on the right side is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for one, the $n = m$ term.

$$\int_0^{2l} \tilde{\phi}(x) \sin \frac{n\pi x}{2l} \, dx = B_n \int_0^{2l} \sin^2 \frac{n\pi x}{2l} \, dx$$

$$= B_n \cdot l$$

The Fourier coefficients are thus

$$B_n = \frac{1}{l} \int_0^{2l} \tilde{\phi}(x) \sin \frac{n\pi x}{2l} \, dx.$$

**Part (c)**

Use the definition of $\tilde{\phi}(x)$ to expand the integral.

$$B_n = \frac{1}{l} \left[ \int_0^l \phi(x) \sin \frac{n\pi x}{2l} \, dx + \int_l^{2l} \phi(2l - x) \sin \frac{n\pi x}{2l} \, dx \right]$$

Substitute $s = x$ in the first integral and $s = 2l - x$ in the second integral.

$$= \frac{1}{l} \left\{ \int_0^l \phi(s) \sin \frac{n\pi s}{2l} \, ds + \int_0^l \phi(s) \sin \left[ \frac{n\pi}{2l} (2l - s) \right] \, (-ds) \right\}$$

$$= \frac{1}{l} \left\{ \int_0^l \phi(s) \sin \frac{n\pi s}{2l} \, ds + \int_0^l \phi(s) \sin \left[ \frac{n\pi}{2l} (2l - s) \right] ds \right\}$$

$$= \frac{1}{l} \int_0^l \phi(s) \left\{ \sin \frac{n\pi s}{2l} + \sin \left[ \frac{n\pi}{2l} (2l - s) \right] \right\} ds$$

Use the sum-to-product formula for sines.

$$= \frac{1}{l} \int_0^l \phi(s) \left\{ 2 \sin \left[ \frac{n\pi s}{2l} + \frac{n\pi}{2l} (2l - s) \right] \cos \left[ \frac{n\pi}{2l} \frac{(2l - s)}{2} \right] \right\} ds$$

$$= \frac{2}{l} \int_0^l \phi(s) \sin \frac{n\pi}{2l} \cos \left[ \frac{n\pi}{2l} (s - l) \right] ds$$

$$= \frac{2}{l} \sin \frac{n\pi}{2l} \int_0^l \phi(s) \cos \left[ \frac{n\pi}{2l} (l - s) \right] ds$$

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Therefore, because of \( \sin(n\pi/2) \), every second coefficient vanishes.

**Part (d)**

The Fourier sine series for \( \tilde{\phi}(x) \) can be simplified (that is, made to converge faster) by summing over the odd values of \( n \) only. Make the substitution \( n = 2k + 1 \) in the series

\[
\tilde{\phi}(x) = \sum_{2k+1=1}^{\infty} B_{2k+1} \sin \left( \frac{(2k+1)\pi x}{2l} \right)
\]

and the corresponding formula for the coefficients.

\[
B_{2k+1} = \frac{2}{l} \int_0^l \phi(s) \sin \left( \frac{(2k+1)\pi s}{2l} \right) ds
\]

\[
= \frac{2}{l} \int_0^l \phi(s) \left[ \cos \left( \frac{(2k+1)\pi s}{2l} \right) + \sin \left( \frac{(2k+1)\pi s}{2l} \right) \right] ds
\]

\[
= \frac{2}{l} (-1)^k \int_0^l \phi(s) \left[ (0) \cos \left( \frac{(2k+1)\pi s}{2l} \right) + (-1)^k \sin \left( \frac{(2k+1)\pi s}{2l} \right) \right] ds
\]

\[
= \frac{2}{l} (-1)^{2k} \int_0^l \phi(s) \sin \left( \frac{(2k+1)\pi s}{2l} \right) ds
\]

\[
= \frac{2}{l} \int_0^l \phi(s) \sin \left( \left( k + \frac{1}{2} \right) \frac{\pi s}{l} \right) ds
\]

Since \( k \) and \( s \) are dummy variables, they can be replaced with \( n \) and \( x \), respectively. In addition, replace \( B_{2k+1} \) with \( C_n \). Then

\[
\tilde{\phi}(x) = \sum_{n=0}^{\infty} C_n \sin \left( \left( n + \frac{1}{2} \right) \frac{\pi x}{l} \right),
\]

where

\[
C_n = \frac{2}{l} \int_0^l \phi(x) \sin \left( \left( n + \frac{1}{2} \right) \frac{\pi x}{l} \right) dx.
\]

On the interval \((0, l)\), \( \tilde{\phi}(x) = \phi(x) \). Therefore,

\[
\phi(x) = \sum_{n=0}^{\infty} C_n \sin \left( \left( n + \frac{1}{2} \right) \frac{\pi x}{l} \right).
\]