Exercise 8

Show directly that $(-X_1'X_2 + X_1X_2')|_a^b = 0$ if both X_1 and X_2 satisfy the same Robin boundary condition at $x = a$ and the same Robin boundary condition at $x = b$.

Solution

X_1 and X_2 are eigenfunctions that satisfy the same Robin boundary conditions at $x = a$ and $x = b$.

$$\begin{aligned} X_1'(a) - A_aX_1(a) &= 0 \qquad & X_2'(a) - A_aX_2(a) &= 0 \\ X_1'(b) + A_bX_1(b) &= 0 \qquad & X_2'(b) + A_bX_2(b) &= 0 \end{aligned}$$

Start with the identity

$$-X_1''X_2 + X_1X_2'' = (-X_1'X_2 + X_1X_2')'.$$

Integrate both sides with respect to x from a to b.

$$\begin{aligned} \int_a^b (-X_1''X_2 + X_1X_2'')\,dx &= (-X_1'X_2 + X_1X_2')\Big|_a^b \\ &= -X_1'(b)X_2(b) + X_1(b)X_2'(b) + X_1'(a)X_2(a) - X_1(a)X_2'(a) \\ &= -[-A_bX_1(b)]X_2(b) + X_1(b)[-A_bX_2(b)] + [A_aX_1(a)]X_2(a) - X_1(a)[A_aX_2(a)] \\ &= \cancel{A_bX_1(b)X_2(b)} - \cancel{A_bX_1(b)X_2(b)} + \cancel{A_aX_1(a)X_2(a)} - \cancel{A_aX_1(a)X_2(a)} \\ &= 0 \end{aligned}$$