Exercise 11

(a) Show that the condition \( f(x)f'(x)\bigg|_{a}^{b} \leq 0 \) is valid for any function \( f(x) \) that satisfies Dirichlet, Neumann, or periodic boundary conditions.

(b) Show that it is also valid for Robin BCs provided that the constants \( a_0 \) and \( a_l \) are positive.

Solution

**Dirichlet Boundary Conditions**

A function \( f(x) \) that satisfies Dirichlet boundary conditions vanishes at both ends of the interval \( a < x < b \).

\[
\begin{align*}
  f(a) &= 0 \\
  f(b) &= 0
\end{align*}
\]

So then

\[
\begin{align*}
  f(x)f'(x)\bigg|_{a}^{b} &= f(b)f'(b) - f(a)f'(a) \\
  &= (0)f'(b) - (0)f'(a) \\
  &= 0.
\end{align*}
\]

Therefore,

\[
f(x)f'(x)\bigg|_{a}^{b} \leq 0
\]

is valid for Dirichlet boundary conditions.

**Neummann Boundary Conditions**

The first derivative of a function \( f(x) \) that satisfies Neumann boundary conditions vanishes at both ends of the interval \( a < x < b \).

\[
\begin{align*}
  f'(a) &= 0 \\
  f'(b) &= 0
\end{align*}
\]

So then

\[
\begin{align*}
  f(x)f'(x)\bigg|_{a}^{b} &= f(b)f'(b) - f(a)f'(a) \\
  &= f(b)(0) - f(a)(0) \\
  &= 0.
\end{align*}
\]

Therefore,

\[
f(x)f'(x)\bigg|_{a}^{b} \leq 0
\]

is valid for Neumann boundary conditions.
**Periodic Boundary Conditions**

A function with periodic boundary conditions has the same value and slope at both ends of the interval $a < x < b$.

\[
\begin{align*}
  f(a) &= f(b) \\
  f'(a) &= f'(b)
\end{align*}
\]

So then

\[
\left. f(x)f'(x) \right|_a^b = f(b)f'(b) - f(a)f'(a)
\]

\[
= f(b)f'(b) - f(b)f'(b)
\]

\[
= 0.
\]

Therefore,

\[
\left. f(x)f'(x) \right|_a^b \leq 0
\]

is valid for periodic boundary conditions.

**Robin Boundary Conditions**

The first derivative of a function that satisfies Robin boundary conditions is proportional to the value of the function at both ends of the interval $a < x < b$.

\[
\begin{align*}
  f'(a) &= a_0 f(a) \\
  f'(b) &= -a_l f(b)
\end{align*}
\]

So then, assuming $a_0$ and $a_l$ are positive,

\[
\left. f(x)f'(x) \right|_a^b = f(b)f'(b) - f(a)f'(a)
\]

\[
= f(b)[-a_l f(b)] - f(a)[a_0 f(a)]
\]

\[
= -a_l[f(b)]^2 - a_0[f(a)]^2
\]

\[
< 0,
\]

since $[f(a)]^2$ and $[f(b)]^2$ are both positive. Therefore,

\[
\left. f(x)f'(x) \right|_a^b \leq 0
\]

is valid for Robin boundary conditions in which $a_0$ and $a_l$ are positive.