Exercise 12

Prove Green’s first identity: For every pair of functions \( f(x), g(x) \) on \((a, b)\),

\[
\int_a^b f''(x)g(x) \, dx = -\int_a^b f'(x)g'(x) \, dx + f'g \bigg|_a^b.
\]

Solution

To solve this problem, one should use integration by parts. The formula for it is

\[
\int_a^b udv = uv \bigg|_a^b - \int_a^b vdu.
\]

Starting from \( \int_a^b f''(x)g(x) \, dx \),

let

\[
\begin{align*}
    u &= g(x) & dv &= f''(x) \, dx \\
    du &= g'(x) \, dx & v &= f'(x).
\end{align*}
\]

Then we have

\[
\int_a^b f''(x)g(x) \, dx = g(x)f'(x) \bigg|_a^b - \int_a^b f'(x)g'(x) \, dx.
\]

Therefore,

\[
\int_a^b f''(x)g(x) \, dx = -\int_a^b f'(x)g'(x) \, dx + f'g \bigg|_a^b.
\]