Exercise 15

Use the same idea as in Exercises 12 and 13 to show that none of the eigenvalues of the fourth-order operator $+d^4/dx^4$ with the boundary conditions $X(0) = X(l) = X''(0) = X''(l) = 0$ are negative.

Solution

The eigenvalue problem for $d^4/dx^4$ is

$$\frac{d^4}{dx^4}X = \lambda X. \quad (1)$$

Consider the integral of $X''''(x)X(x)$ with respect to $x$ over the interval $(0, l)$ and use integration by parts.

$$\int_0^l X''''X \, dx = X''''|_0^l - \int_0^l X'''X' \, dx$$

$$= X''''(l)X(l) - X''''(0)X(0) - \int_0^l X'''X' \, dx$$

$X(l) = 0$ and $X(0) = 0$, so the first two terms vanish.

$$= - \int_0^l X'''X' \, dx$$

Use integration by parts once more.

$$= - \left[ X''X'|_0^l - \int_0^l X''^2 \, dx \right]$$

$$= - \left[ X''(l)X'(l) - X''(0)X'(0) - \int_0^l X''^2 \, dx \right]$$

$X''(l) = 0$ and $X''(0) = 0$, so the first two terms vanish.

$$= \int_0^l X''^2 \, dx$$

Substitute equation (1) into the left side.

$$\int_0^l (\lambda X)X \, dx = \int_0^l X''^2 \, dx$$

$$\int_0^l \lambda X^2 \, dx = \int_0^l X''^2 \, dx$$

$\lambda$ has to be positive because $X^2$ and $X''^2$ are positive. Therefore, none of the eigenvalues are negative.