Exercise 2

(a) On the interval $[-1, 1]$, show that the function x is orthogonal to the constant functions.

(b) Find a quadratic polynomial that is orthogonal to both 1 and x.

(c) Find a cubic polynomial that is orthogonal to all quadratics. (These are the first few *Legendre polynomials.*)

Solution

A function $f(x)$ is said to be orthogonal to another one $g(x)$ on the interval (a, b) if

$$\int_a^b f(x)g(x)\,dx = 0.$$

Part (a)

Here $f(x) = x$ and $g(x) = C$.

$$\begin{aligned}\int_{-1}^1 f(x)g(x)\,dx &= \int_{-1}^1 Cx\,dx \\ &= C \cdot \frac{x^2}{2}\bigg|_{-1}^1 \\ &= C\left(\frac{1^2}{2} - \frac{(-1)^2}{2}\right) \\ &= 0\end{aligned}$$

Therefore, the function x is orthogonal to the constant functions.

Part (b)

We will set $f(x) = ax^2 + bx + c$ and use the fact that it is orthogonal to both 1 and x to determine a, b, and c.

$$\begin{aligned}\int_{-1}^1 (ax^2 + bx + c)(1)\,dx &= 0 \\ \int_{-1}^1 (ax^2 + bx + c)\,dx &= 0 \\ a\int_{-1}^1 x^2\,dx + b\int_{-1}^1 x\,dx + c\int_{-1}^1 dx &= 0 \\ a \cdot \frac{x^3}{3}\bigg|_{-1}^1 + b \cdot \frac{x^2}{2}\bigg|_{-1}^1 + c \cdot \frac{x}{1}\bigg|_{-1}^1 &= 0 \\ a\left(\frac{1}{3} + \frac{1}{3}\right) + b\left(\frac{1}{2} - \frac{1}{2}\right) + c(1+1) &= 0 \\ \frac{2}{3}a + 2c &= 0 \\ c &= -\frac{1}{3}a\end{aligned}$$

$$\begin{aligned}\int_{-1}^1 (ax^2 + bx + c)(x)\,dx &= 0 \\ \int_{-1}^1 (ax^3 + bx^2 + cx)\,dx &= 0 \\ a\int_{-1}^1 x^3\,dx + b\int_{-1}^1 x^2\,dx + c\int_{-1}^1 x\,dx &= 0 \\ a \cdot \frac{x^4}{4}\bigg|_{-1}^1 + b \cdot \frac{x^3}{3}\bigg|_{-1}^1 + c \cdot \frac{x^2}{2}\bigg|_{-1}^1 &= 0 \\ a\left(\frac{1}{4} - \frac{1}{4}\right) + b\left(\frac{1}{3} + \frac{1}{3}\right) + c\left(\frac{1}{2} - \frac{1}{2}\right) &= 0 \\ \frac{2}{3}b &= 0 \\ b &= 0\end{aligned}$$

Therefore, the quadratic polynomial orthogonal to 1 and x is

$$f(x) = ax^2 - \frac{1}{3}a,$$

that is, any multiple of $x^2 - 1/3$. Setting $a = 3$ gives us the answer at the back of the book,

$$f(x) = 3x^2 - 1.$$

Part (c)

We will set $f(x) = Ax^3 + Bx^2 + Cx + D$ and $g(x) = kx^2 + lx + m$ and use the fact that they are orthogonal to determine conditions for a, b, c, and d.

$$\int_{-1}^{1} (Ax^3 + Bx^2 + Cx + D)(kx^2 + lx + m)\, dx = 0$$

Expand the integrand and then factor it in powers of x.

$$\int_{-1}^{1} [Akx^5 + (Al + Bk)x^4 + (Am + Bl + Ck)x^3 + (Bm + Cl + Dk)x^2 + (Cm + Dl)x + Dm]\, dx = 0$$

Split up the integral into six and bring the constants in front of them.

$$Ak \int_{-1}^{1} x^5\, dx + (Al + Bk) \int_{-1}^{1} x^4\, dx + (Am + Bl + Ck) \int_{-1}^{1} x^3\, dx \\ + (Bm + Cl + Dk) \int_{-1}^{1} x^2\, dx + (Cm + Dl) \int_{-1}^{1} x\, dx + Dm \int_{-1}^{1} dx = 0$$

Evaluate the integrals.

$$Ak \cdot \frac{x^6}{6}\bigg|_{-1}^{1} + (Al + Bk) \cdot \frac{x^5}{5}\bigg|_{-1}^{1} + (Am + Bl + Ck) \cdot \frac{x^4}{4}\bigg|_{-1}^{1} \\ + (Bm + Cl + Dk) \cdot \frac{x^3}{3}\bigg|_{-1}^{1} + (Cm + Dl) \cdot \frac{x^2}{2}\bigg|_{-1}^{1} + Dm \cdot \frac{x}{1}\bigg|_{-1}^{1} = 0$$

$$Ak\left(\frac{1}{6} - \frac{1}{6}\right) + (Al + Bk)\left(\frac{1}{5} + \frac{1}{5}\right) + (Am + Bl + Ck)\left(\frac{1}{4} - \frac{1}{4}\right) \\ + (Bm + Cl + Dk)\left(\frac{1}{3} + \frac{1}{3}\right) + (Cm + Dl)\left(\frac{1}{2} - \frac{1}{2}\right) + Dm(1 + 1) = 0$$

$$\frac{2}{5}(Al + Bk) + \frac{2}{3}(Bm + Cl + Dk) + 2Dm = 0$$

Expand the left side and then factor it in k, l, and m.

$$\left(\frac{2B}{5} + \frac{2D}{3}\right)k + \left(\frac{2A}{5} + \frac{2C}{3}\right)l + \left(\frac{2B}{3} + 2D\right)m = 0$$

Since k, l, and m are arbitrary, their coefficients must be equal to zero for the equation to be satisfied.

$$\frac{2B}{5} + \frac{2D}{3} = 0$$
$$\frac{2A}{5} + \frac{2C}{3} = 0$$
$$\frac{2B}{3} + 2D = 0$$

The first and third equations imply that $B = 0$ and $D = 0$. Solve the second equation for C.

$$C = -\frac{3}{5}A$$

The cubic polynomial orthogonal to all quadratics is

$$f(x) = Ax^3 - \frac{3}{5}Ax,$$

that is, any multiple of $x^3 - 3x/5$. Setting $A = 5/2$ gives us

$$f(x) = \frac{1}{2}(5x^3 - 3x),$$

which is the third Legendre polynomial $P_3(x)$.