**Exercise 6**

Find the complex eigenvalues of the first-derivative operator $\frac{d}{dx}$ subject to the single boundary condition $X(0) = X(1)$. Are the eigenfunctions orthogonal on the interval $(0, 1)$?

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**Solution**

The eigenvalue problem for $\frac{d}{dx}$ is

$$\frac{d}{dx} X = \lambda X.$$ 

Solve this for $X$ by separating variables.

$$\frac{dX}{X} = \lambda \, dx$$

Integrate both sides.

$$\int \frac{dX}{X} = \int \lambda \, dx$$

Evaluate the integrals.

$$\ln |X| = \lambda x + C$$

Exponentiate both sides.

$$|X| = e^{\lambda x + C}$$

$$= e^{\lambda x} e^C$$

Introduce $\pm$ on the right side to remove the absolute value sign.

$$X(x) = \pm e^C e^{\lambda x}$$

Use a new constant of integration.

$$X(x) = Ae^{\lambda x}$$

$$= Ae^{-i^2 \lambda x}$$

$$= Ae^{-i(i\lambda x)}$$

$$= A(\cos i\lambda x - i \sin i\lambda x)$$

The boundary condition $X(0) = X(1)$ implies that

$$A = A(\cos i\lambda - i \sin i\lambda),$$

so the equation to solve for the eigenvalues is

$$\cos i\lambda - i \sin i\lambda = 1.$$ 

Matching the real and imaginary parts of both sides of the equation, we have

$$\cos i\lambda = 1 \quad \text{and} \quad \sin i\lambda = 0.$$
To satisfy \( \sin i\lambda = 0 \), \( i\lambda \) has to be an integer multiple of \( \pi \), but to satisfy \( \cos i\lambda = 1 \) as well, \( i\lambda \) has to be an even multiple of \( \pi \). Therefore, the complex eigenvalues of \( d/dx \) are

\[
\lambda = \lambda_n = 2n\pi i, \quad n = 0, \pm 1, \ldots
\]

The eigenfunctions associated with these eigenvalues are

\[
X(x) = Ae^{\lambda x} \rightarrow X_n(x) = e^{2n\pi ix}.
\]

Now we will check to see whether or not they are orthogonal on the interval \((0, 1)\). Since the eigenfunctions are complex, we have to calculate the inner product,

\[
\int_0^1 X_n(x)X_m(x)dx,
\]

where \( n \) and \( m \) are distinct integers. If the integral is equal to zero, then the eigenfunctions are in fact orthogonal over the interval.

\[
\begin{align*}
\int_0^1 X_n(x)X_m(x)dx &= \int_0^1 e^{2n\pi ix}e^{-2m\pi ix} \, dx \\
&= \int_0^1 e^{2(n-m)\pi ix} \, dx \\
&= \left. \frac{1}{2(n-m)\pi i} e^{2(n-m)\pi ix} \right|_0^1 \\
&= \frac{1}{2(n-m)\pi i} (e^{2(n-m)\pi i} - 1) \\
&= \frac{1}{2(n-m)\pi i} (\cos[2(n-m)\pi] + i\sin[2(n-m)\pi] - 1) \\
&= 0
\end{align*}
\]

Therefore, the eigenfunctions are orthogonal on the interval \((0, 1)\).