# Exercise 6

Find the complex eigenvalues of the first-derivative operator $d/dx$ subject to the single boundary condition $X(0) = X(1)$. Are the eigenfunctions orthogonal on the interval $(0, 1)$?

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**Solution**

The eigenvalue problem for $d/dx$ is

$$\frac{d}{dx}X = \lambda X.$$

Solve this for $X$ by separating variables.

$$\frac{dX}{X} = \lambda\, dx$$

Integrate both sides.

$$\int \frac{dX}{X} = \int \lambda\, dx$$

Evaluate the integrals.

$$\ln|X| = \lambda x + C$$

Exponentiate both sides.

$$\begin{aligned} |X| &= e^{\lambda x + C} \\ &= e^{\lambda x} e^{C} \end{aligned}$$

Introduce $\pm$ on the right side to remove the absolute value sign.

$$X(x) = \pm e^{C} e^{\lambda x}$$

Use a new constant of integration.

$$\begin{aligned} X(x) &= Ae^{\lambda x} \\ &= Ae^{-i^2 \lambda x} \\ &= Ae^{-i(i\lambda x)} \\ &= A(\cos i\lambda x - i \sin i\lambda x) \end{aligned}$$

The boundary condition $X(0) = X(1)$ implies that

$$A = A(\cos i\lambda - i \sin i\lambda),$$

so the equation to solve for the eigenvalues is

$$\cos i\lambda - i \sin i\lambda = 1.$$

Matching the real and imaginary parts of both sides of the equation, we have

$$\cos i\lambda = 1 \quad \text{and} \quad \sin i\lambda = 0.$$

To satisfy $\sin i\lambda = 0$, $i\lambda$ has to be an integer multiple of $\pi$, but to satisfy $\cos i\lambda = 1$ as well, $i\lambda$ has to be an even multiple of $\pi$. Therefore, the complex eigenvalues of $d/dx$ are

$$\lambda = \lambda_n = 2n\pi i, \quad n = 0, \pm 1, \ldots.$$

The eigenfunctions associated with these eigenvalues are

$$X(x) = Ae^{\lambda x} \quad \rightarrow \quad X_n(x) = e^{2n\pi ix}.$$

Now we will check to see whether or not they are orthogonal on the interval $(0, 1)$. Since the eigenfunctions are complex, we have to calculate the inner product,

$$\int_0^1 X_n(x)\overline{X_m(x)}\,dx,$$

where $n$ and $m$ are distinct integers. If the integral is equal to zero, then the eigenfunctions are in fact orthogonal over the interval.

$$\begin{aligned}
\int_0^1 X_n(x)\overline{X_m(x)}\,dx &= \int_0^1 e^{2n\pi ix}e^{-2m\pi ix}\,dx \\
&= \int_0^1 e^{2(n-m)\pi ix}\,dx \\
&= \frac{1}{2(n-m)\pi i}e^{2(n-m)\pi ix}\Big|_0^1 \\
&= \frac{1}{2(n-m)\pi i}(e^{2(n-m)\pi i} - 1) \\
&= \frac{1}{2(n-m)\pi i}(\underbrace{\cos[2(n-m)\pi]}_{=1} + i\underbrace{\sin[2(n-m)\pi]}_{=0} - 1) \\
&= 0
\end{aligned}$$

Therefore, the eigenfunctions are orthogonal on the interval $(0, 1)$.