# Exercise 7

Show *by direct integration* that the eigenfunctions associated with the Robin BCs, namely,

$$\phi_n(x) = \cos\beta_n x + \frac{a_0}{\beta_n}\sin\beta_n x \quad \text{where } \lambda_n = \beta_n^2,$$

are mutually orthogonal on $0 \leq x \leq l$, where $\beta_n$ are the positive roots of (4.3.8).

---

### Solution

For the eigenfunctions to be mutually orthogonal on $0 \leq x \leq l$, their inner product must satisfy

$$\int_0^l \phi_n(x)\phi_m(x)\,dx = 0,$$

where $n$ and $m$ are distinct integers. Our aim then is to evaluate the integral.

$$\begin{aligned}
\int_0^l \phi_n(x)\phi_m(x)\,dx &= \int_0^l \left(\cos\beta_n x + \frac{a_0}{\beta_n}\sin\beta_n x\right)\left(\cos\beta_m x + \frac{a_0}{\beta_m}\sin\beta_m x\right)dx \\
&= \int_0^l \left(\cos\beta_n x\cos\beta_m x + \frac{a_0}{\beta_m}\cos\beta_n x\sin\beta_m x + \frac{a_0}{\beta_n}\sin\beta_n x\cos\beta_m x \right. \\
&\qquad\qquad \left. + \frac{a_0^2}{\beta_n\beta_m}\sin\beta_n x\sin\beta_m x\right)dx \\
&= \int_0^l \cos\beta_n x\cos\beta_m x\,dx + \frac{a_0}{\beta_m}\int_0^l \cos\beta_n x\sin\beta_m x\,dx \\
&\quad + \frac{a_0}{\beta_n}\int_0^l \sin\beta_n x\cos\beta_m x\,dx + \frac{a_0^2}{\beta_n\beta_m}\int_0^l \sin\beta_n x\sin\beta_m x\,dx \\
&= \int_0^l \frac{1}{2}[\cos(\beta_n x - \beta_m x) + \cos(\beta_n x + \beta_m x)]\,dx \\
&\quad + \frac{a_0}{\beta_m}\int_0^l \frac{1}{2}[\sin(\beta_n x + \beta_m x) - \sin(\beta_n x - \beta_m x)]\,dx \\
&\quad + \frac{a_0}{\beta_n}\int_0^l \frac{1}{2}[\sin(\beta_n x + \beta_m x) + \sin(\beta_n x - \beta_m x)]\,dx \\
&\quad + \frac{a_0^2}{\beta_n\beta_m}\int_0^l \frac{1}{2}[\cos(\beta_n x - \beta_m x) - \cos(\beta_n x + \beta_m x)]\,dx \\
&= \frac{1}{2}\left\{\int_0^l \cos[(\beta_n - \beta_m)x]\,dx + \int_0^l \cos[(\beta_n + \beta_m)x]\,dx\right\} \\
&\quad + \frac{a_0}{2\beta_m}\left\{\int_0^l \sin[(\beta_n + \beta_m)x]\,dx - \int_0^l \sin[(\beta_n - \beta_m)x]\,dx\right\} \\
&\quad + \frac{a_0}{2\beta_n}\left\{\int_0^l \sin[(\beta_n + \beta_m)x]\,dx + \int_0^l \sin[(\beta_n - \beta_m)x]\,dx\right\} \\
&\quad + \frac{a_0^2}{2\beta_n\beta_m}\left\{\int_0^l \cos[(\beta_n - \beta_m)x]\,dx - \int_0^l \cos[(\beta_n + \beta_m)x]\,dx\right\}
\end{aligned}$$

$$
\begin{aligned}
\int_0^l \phi_n(x)\phi_m(x)\,dx &= \frac{1}{2}\left\{ \frac{1}{\beta_n-\beta_m}\sin[(\beta_n-\beta_m)x]\Big|_0^l + \frac{1}{\beta_n+\beta_m}\sin[(\beta_n+\beta_m)x]\Big|_0^l \right\} \\
&\quad + \frac{a_0}{2\beta_m}\left\{ -\frac{1}{\beta_n+\beta_m}\cos[(\beta_n+\beta_m)x]\Big|_0^l + \frac{1}{\beta_n-\beta_m}\cos[(\beta_n-\beta_m)x]\Big|_0^l \right\} \\
&\quad + \frac{a_0}{2\beta_n}\left\{ -\frac{1}{\beta_n+\beta_m}\cos[(\beta_n+\beta_m)x]\Big|_0^l - \frac{1}{\beta_n-\beta_m}\cos[(\beta_n-\beta_m)x]\Big|_0^l \right\} \\
&\quad + \frac{a_0^2}{2\beta_n\beta_m}\left\{ \frac{1}{\beta_n-\beta_m}\sin[(\beta_n-\beta_m)x]\Big|_0^l - \frac{1}{\beta_n+\beta_m}\sin[(\beta_n+\beta_m)x]\Big|_0^l \right\} \\
&= \frac{1}{2}\left\{ \frac{1}{\beta_n-\beta_m}\sin[(\beta_n-\beta_m)l] + \frac{1}{\beta_n+\beta_m}\sin[(\beta_n+\beta_m)l] \right\} \\
&\quad + \frac{a_0}{2\beta_m}\left\{ -\frac{\cos[(\beta_n+\beta_m)l]}{\beta_n+\beta_m} + \frac{1}{\beta_n+\beta_m} + \frac{\cos[(\beta_n-\beta_m)l]}{\beta_n-\beta_m} - \frac{1}{\beta_n-\beta_m} \right\} \\
&\quad + \frac{a_0}{2\beta_n}\left\{ -\frac{\cos[(\beta_n+\beta_m)l]}{\beta_n+\beta_m} + \frac{1}{\beta_n+\beta_m} - \frac{\cos[(\beta_n-\beta_m)l]}{\beta_n-\beta_m} + \frac{1}{\beta_n-\beta_m} \right\} \\
&\quad + \frac{a_0^2}{2\beta_n\beta_m}\left\{ \frac{1}{\beta_n-\beta_m}\sin[(\beta_n-\beta_m)l] - \frac{1}{\beta_n+\beta_m}\sin[(\beta_n+\beta_m)l] \right\} \\
&= \frac{1}{2}\left\{ \frac{(\beta_n+\beta_m)\sin[(\beta_n-\beta_m)l] + (\beta_n-\beta_m)\sin[(\beta_n+\beta_m)l]}{\beta_n^2-\beta_m^2} \right\} \\
&\quad + \frac{a_0}{2\beta_m}\left\{ \frac{(\beta_n+\beta_m)\cos[(\beta_n-\beta_m)l] - (\beta_n-\beta_m)\cos[(\beta_n+\beta_m)l]}{\beta_n^2-\beta_m^2} - \cancel{\frac{2\beta_m}{\beta_n^2-\beta_m^2}} \right\} \\
&\quad + \frac{a_0}{2\beta_n}\left\{ -\frac{(\beta_n-\beta_m)\cos[(\beta_n+\beta_m)l] + (\beta_n+\beta_m)\cos[(\beta_n-\beta_m)l]}{\beta_n^2-\beta_m^2} + \cancel{\frac{2\beta_n}{\beta_n^2-\beta_m^2}} \right\} \\
&\quad + \frac{a_0^2}{2\beta_n\beta_m}\left\{ \frac{(\beta_n+\beta_m)\sin[(\beta_n-\beta_m)l] - (\beta_n-\beta_m)\sin[(\beta_n+\beta_m)l]}{\beta_n^2-\beta_m^2} \right\} \\
&= \frac{1}{2(\beta_n^2-\beta_m^2)}\left\{ (\beta_n+\beta_m)\left(1+\frac{a_0^2}{\beta_n\beta_m}\right)\sin[(\beta_n-\beta_m)l] \right. \\
&\quad + (\beta_n-\beta_m)\left(1-\frac{a_0^2}{\beta_n\beta_m}\right)\sin[(\beta_n+\beta_m)l] \\
&\quad + a_0(\beta_n+\beta_m)\left(\frac{1}{\beta_m}-\frac{1}{\beta_n}\right)\cos[(\beta_n-\beta_m)l] \\
&\quad \left. - a_0(\beta_n-\beta_m)\left(\frac{1}{\beta_m}+\frac{1}{\beta_n}\right)\cos[(\beta_n+\beta_m)l] \right\} \\
&= \frac{1}{2(\beta_n^2-\beta_m^2)}\left[ \frac{\beta_n+\beta_m}{\beta_n\beta_m}\left(\beta_n\beta_m+a_0^2\right)(\sin\beta_n l\cos\beta_m l - \cos\beta_n l\sin\beta_m l) \right. \\
&\quad + \frac{\beta_n-\beta_m}{\beta_n\beta_m}\left(\beta_n\beta_m-a_0^2\right)(\sin\beta_n l\cos\beta_m l + \cos\beta_n l\sin\beta_m l) \\
&\quad + a_0\frac{\beta_n^2-\beta_m^2}{\beta_n\beta_m}(\cancel{\cos\beta_n l\cos\beta_m l} + \sin\beta_n l\sin\beta_m l) \\
&\quad \left. - a_0\frac{\beta_n^2-\beta_m^2}{\beta_n\beta_m}(\cancel{\cos\beta_n l\cos\beta_m l} - \sin\beta_n l\sin\beta_m l) \right]
\end{aligned}
$$

$$\begin{aligned}
\int_0^l \phi_n(x)\phi_m(x)\,dx &= \frac{1}{2\beta_n\beta_m(\beta_n^2-\beta_m^2)}\bigg[(\beta_n^2\beta_m+\beta_n\beta_m^2+\beta_n a_0^2+\beta_m a_0^2)(\sin\beta_n l\cos\beta_m l-\cos\beta_n l\sin\beta_m l) \\
&\qquad + (\beta_n^2\beta_m-\beta_n\beta_m^2-\beta_n a_0^2+\beta_m a_0^2)(\sin\beta_n l\cos\beta_m l+\cos\beta_n l\sin\beta_m l) \\
&\qquad + 2a_0(\beta_n^2-\beta_m^2)\sin\beta_n l\sin\beta_m l\bigg] \\
&= \frac{1}{2\beta_n\beta_m(\beta_n^2-\beta_m^2)}\bigg[2\beta_n^2\beta_m\sin\beta_n l\cos\beta_m l-2\beta_n\beta_m^2\cos\beta_n l\sin\beta_m l \\
&\qquad - 2\beta_n a_0^2\cos\beta_n l\sin\beta_m l+2\beta_m a_0^2\sin\beta_n l\cos\beta_m l \\
&\qquad + 2a_0(\beta_n^2-\beta_m^2)\sin\beta_n l\sin\beta_m l\bigg] \\
&= \frac{1}{\beta_n\beta_m(\beta_n^2-\beta_m^2)}\bigg[\beta_m(\beta_n^2+a_0^2)\sin\beta_n l\cos\beta_m l-\beta_n(\beta_m^2+a_0^2)\cos\beta_n l\sin\beta_m l \\
&\qquad + a_0(\beta_n^2-\beta_m^2)\sin\beta_n l\sin\beta_m l\bigg] \\
&= \frac{\cos\beta_n l\cos\beta_m l}{\beta_n\beta_m(\beta_n^2-\beta_m^2)}\bigg[\beta_m(\beta_n^2+a_0^2)\tan\beta_n l-\beta_n(\beta_m^2+a_0^2)\tan\beta_m l \\
&\qquad + a_0(\beta_n^2-\beta_m^2)\tan\beta_n l\tan\beta_m l\bigg]
\end{aligned}$$

Equation (4.3.8) on page 93 in the textbook says that $(\beta^2-a_0a_l)\tan\beta l=(a_0+a_l)\beta$.

$$\begin{aligned}
&= \frac{\cos\beta_n l\cos\beta_m l}{\beta_n\beta_m(\beta_n^2-\beta_m^2)}\bigg[\beta_m(\beta_n^2+a_0^2)\frac{(a_0+a_l)\beta_n}{\beta_n^2-a_0a_l}-\beta_n(\beta_m^2+a_0^2)\frac{(a_0+a_l)\beta_m}{\beta_m^2-a_0a_l} \\
&\qquad + a_0(\beta_n^2-\beta_m^2)\frac{(a_0+a_l)\beta_n}{\beta_n^2-a_0a_l}\frac{(a_0+a_l)\beta_m}{\beta_m^2-a_0a_l}\bigg] \\
&= \frac{(a_0+a_l)\cos\beta_n l\cos\beta_m l}{(\beta_n^2-\beta_m^2)(\beta_n^2-a_0a_l)(\beta_m^2-a_0a_l)}\bigg[(\beta_n^2+a_0^2)(\beta_m^2-a_0a_l)-(\beta_m^2+a_0^2)(\beta_n^2-a_0a_l) \\
&\qquad + a_0(\beta_n^2-\beta_m^2)(a_0+a_l)\bigg] \\
&= \frac{(a_0+a_l)\cos\beta_n l\cos\beta_m l}{(\beta_n^2-\beta_m^2)(\beta_n^2-a_0a_l)(\beta_m^2-a_0a_l)}\bigg(\cancel{\beta_n^2\beta_m^2}-a_0a_l\beta_n^2+a_0^2\beta_m^2-\cancel{a_0^3a_l} \\
&\qquad -\cancel{\beta_n^2\beta_m^2}+\cancel{a_0a_l\beta_m^2}-a_0^2\beta_n^2+\cancel{a_0^3a_l} \\
&\qquad +a_0^2\beta_n^2+a_0a_l\beta_n^2-a_0^2\beta_m^2-\cancel{a_0a_l\beta_m^2}\bigg) \\
&= \frac{(a_0+a_l)\cos\beta_n l\cos\beta_m l}{(\beta_n^2-\beta_m^2)(\beta_n^2-a_0a_l)(\beta_m^2-a_0a_l)}\bigg(-\cancel{a_0a_l\beta_n^2}+\cancel{a_0^2\beta_m^2}-\cancel{a_0^2\beta_n^2} \\
&\qquad +\cancel{a_0^2\beta_n^2}+\cancel{a_0a_l\beta_n^2}-\cancel{a_0^2\beta_m^2}\bigg) \\
&= 0
\end{aligned}$$

Therefore, the eigenfunctions associated with the Robin BCs are mutually orthogonal on $0\le x\le l$.