Exercise 3

Prove the inequality $l\int_0^l (f'(x))^2\,dx \geq [f(l)-f(0)]^2$ for any real function $f(x)$ whose derivative $f'(x)$ is continuous. [*Hint:* Use Schwarz's inequality with the pair $f'(x)$ and 1.]

Solution

The Schwarz inequality states that for any two functions, $g(x)$ and $h(x)$,

$$\|g\|\|h\| \geq |(g,h)|.$$

Here we let $g(x) = f'(x)$ and $h(x) = 1$. Then

$$\begin{gathered}
\|f'\|\|1\| \geq |(f',1)| \\
\|f'\|^2\|1\|^2 \geq |(f',1)|^2 \\
\int_0^l |f'(x)|^2\,dx \int_0^l |1|^2\,dx \geq \left[\int_0^l f'(x)\cdot 1\,dx\right]^2 \\
\int_0^l [f'(x)]^2\,dx \int_0^l dx \geq \left[\int_0^l f'(x)\,dx\right]^2 \\
\int_0^l [f'(x)]^2\,dx(l) \geq \left[f(x)\Big|_0^l\right]^2.
\end{gathered}$$

Therefore,

$$l\int_0^l [f'(x)]^2\,dx \geq [f(l)-f(0)]^2.$$