Exercise 3

Prove the inequality $l \int_0^l (f'(x))^2 \, dx \geq [f(l) - f(0)]^2$ for any real function $f(x)$ whose derivative $f'(x)$ is continuous. [Hint: Use Schwarz’s inequality with the pair $f'(x)$ and 1.]

Solution

The Schwarz inequality states that for any two functions, $g(x)$ and $h(x)$,

$$\|g\|\|h\| \geq |(g, h)|.$$

Here we let $g(x) = f'(x)$ and $h(x) = 1$. Then

$$\|f'\|\|1\| \geq |(f', 1)|$$

$$\|f'\|^2\|1\|^2 \geq |(f', 1)|^2$$

$$\int_0^l |f'(x)|^2 \, dx \int_0^l 1^2 \, dx \geq \left[ \int_0^l f'(x) \cdot 1 \, dx \right]^2$$

$$\int_0^l |f'(x)|^2 \, dx \int_0^l 1^2 \, dx \geq \left[ \int_0^l f'(x) \, dx \right]^2$$

$$\int_0^l |f'(x)|^2 \, dx \cdot 1(l) \geq \left[ f(x) \bigg|_0^l \right]^2.$$

Therefore,

$$l \int_0^l [f'(x)]^2 \, dx \geq [f(l) - f(0)]^2.$$