Exercise 4

(a) Solve the problem $u_t = ku_{xx}$ for $0 < x < l$, $u(x,0) = \phi(x)$, with the unusual boundary conditions

$$u_x(0,t) = u_x(l,t) = \frac{u(l,t) - u(0,t)}{l}.$$

Assume that there are no negative eigenvalues. (*Hint:* See Exercise 4.3.12.)

(b) Show that as $t \to \infty$,

$$\lim u(x,t) = A + Bx,$$

assuming that you can take limits term by term.

(c) Use Green's first identity and Exercise 3 to show that there are no negative eigenvalues.

(d) Find A and B. (*Hint:* $A + Bx$ is the beginning of the series. Take the inner product of the series for $\phi(x)$ with each of the functions 1 and x. Make use of the orthogonality.)

Solution

See the solution to Exercise 4.3.12 for the answers to (a), (b), and (d). It will be shown here that there are no negative eigenvalues. Green's first identity says that for any two functions, $f(x)$ and $g(x)$,

$$\int_a^b f''(x)g(x)\,dx = -\int_a^b f'(x)g'(x)\,dx + f'g\bigg|_a^b.$$

Set f and g equal to $u(x,t)$.

$$\begin{aligned}
\int_0^l u_{xx}u\,dx &= -\int_0^l u_x^2\,dx + u_xu\bigg|_0^l \\
&= -\int_0^l u_x^2\,dx + u_x(l,t)u(l,t) - u_x(0,t)u(0,t) \\
&= -\int_0^l u_x^2\,dx + u_x(l,t)[u(l,t) - u(0,t)] \\
&= -\int_0^l u_x^2\,dx + \frac{[u(l,t) - u(0,t)]^2}{l}
\end{aligned}$$

From Exercise 3, we have

$$l\int_0^l [f'(x)]^2\,dx \geq [f(l) - f(0)]^2.$$

Set f equal to $u(x,t)$ and divide both sides by l.

$$\int_0^l u_x^2\,dx \geq \frac{[u(l,t) - u(0,t)]^2}{l}$$

We then conclude from Green's first identity that

$$\int_0^l u_{xx}u\,dx \leq 0$$

for all t. When we separate variables, $u = X(x)T(t)$, and

$$\int_0^l X''(x)T(t)X(x)T(t)\,dx \leq 0.$$

Bring $[T(t)]^2$ in front of the integral.

$$[T(t)]^2 \int_0^l X''(x)X(x)\,dx \leq 0$$

The eigenvalue problem that comes about from separating variables is $X'' = -\lambda X$.

$$[T(t)]^2 \int_0^l (-\lambda)[X(x)]^2\,dx \leq 0$$

The only way this inequality is satisfied is if λ is zero or positive, as $[X(x)]^2$ and $[T(t)]^2$ are both positive. Therefore, there are no negative eigenvalues.