Exercise 15

Carry out the step going from (24) to (25).

Solution

$$u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} A_n \left(\sin\frac{n\pi(x+ct)}{l} + \sin\frac{n\pi(x-ct)}{l}\right) + \frac{1}{2c}\sum_{n=1}^{\infty}\int_{x-ct}^{x+ct} B_n \frac{n\pi c}{l}\sin\frac{n\pi s}{l}\,ds \tag{24}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} A_n \left[\sin\left(\frac{n\pi x}{l}+\frac{n\pi ct}{l}\right) + \sin\left(\frac{n\pi x}{l}-\frac{n\pi ct}{l}\right)\right] + \frac{1}{2c}\sum_{n=1}^{\infty} B_n \frac{n\pi c}{l}\int_{x-ct}^{x+ct}\sin\frac{n\pi s}{l}\,ds$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} A_n \left[\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right) + \cancel{\cos\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi ct}{l}\right)}\right.$$
$$\left. + \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right) - \cancel{\cos\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi ct}{l}\right)}\right]$$
$$+ \frac{1}{2c}\sum_{n=1}^{\infty} B_n \frac{n\pi c}{l}\left(-\frac{l}{n\pi}\right)\cos\frac{n\pi s}{l}\Bigg|_{x-ct}^{x+ct}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} A_n \left[2\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right)\right] - \frac{1}{2}\sum_{n=1}^{\infty} B_n \left[\cos\frac{n\pi(x+ct)}{l} - \cos\frac{n\pi(x-ct)}{l}\right]$$

$$= \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} - \frac{1}{2}\sum_{n=1}^{\infty} B_n \left[\cos\left(\frac{n\pi x}{l}+\frac{n\pi ct}{l}\right) - \cos\left(\frac{n\pi x}{l}-\frac{n\pi ct}{l}\right)\right]$$

$$= \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}$$
$$- \frac{1}{2}\sum_{n=1}^{\infty} B_n \left\{\cancel{\cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right)} - \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi ct}{l}\right)\right.$$
$$\left. - \left[\cancel{\cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right)} + \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi ct}{l}\right)\right]\right\}$$

$$= \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} - \frac{1}{2}\sum_{n=1}^{\infty} B_n \left[-2\sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi ct}{l}\right)\right]$$

$$= \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\sin\frac{n\pi ct}{l}$$

$$= \sum_{n=1}^{\infty}\left(A_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi x}{l}\sin\frac{n\pi ct}{l}\right)$$

$$= \sum_{n}\left(A_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi x}{l}\sin\frac{n\pi ct}{l}\right) \tag{25}$$