Exercise 5

Solve $u_{tt} = c^2 u_{xx} + e^t \sin 5x$ for $0 < x < \pi$, with $u(0,t) = u(\pi,t) = 0$ and the initial conditions $u(x,0) = 0$, $u_t(x,0) = \sin 3x$.

Solution

Since the PDE is linear and inhomogeneous, we choose to apply the method of eigenfunction expansion to solve it. Consider the eigenvalue problem of the differential operator involving the spatial variable x

$$\frac{d^2}{dx^2}\phi = \lambda\phi \tag{1}$$

with the same boundary conditions as u.

$$\phi(0) = 0$$
$$\phi(\pi) = 0$$

Values of λ for which the boundary conditions are satisfied are known as the eigenvalues, and the nontrivial solutions associated with them are called the eigenfunctions. Equation (1) is known as the one-dimensional Helmholtz equation; the eigenfunctions for it are known to be orthogonal and form a complete set, which will prove useful later.

Determination of Positive Eigenvalues: $\lambda = \mu^2$

Suppose that λ is positive. Then equation (1) becomes

$$\frac{d^2\phi}{dx^2} = \mu^2\phi.$$

Its solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$\phi(x) = C_1 \cosh \mu x + C_2 \sinh \mu x$$

Apply the boundary conditions to determine C_1 and C_2.

$$\phi(0) = C_1 = 0$$
$$\phi(\pi) = C_1 \cosh \mu\pi + C_2 \sinh \mu\pi = 0$$

Since $C_1 = 0$, the second equation reduces to $C_2 \sinh \mu\pi = 0$. Hyperbolic sine is not oscillatory, so the only way this equation is satisfied is if $C_2 = 0$. The trivial solution is obtained, so there are no positive eigenvalues.

Determination of the Zero Eigenvalue: $\lambda = 0$

Suppose that λ is zero. Then equation (1) becomes

$$\frac{d^2\phi}{dx^2} = 0.$$

The general solution is obtained by integrating both sides with respect to x twice.

$$\phi(x) = C_3 x + C_4$$

Apply the boundary conditions to determine C_3 and C_4.

$$\begin{aligned}\phi(0) &= C_4 = 0\\ \phi(\pi) &= C_3\pi + C_4 = 0\end{aligned}$$

Since $C_4 = 0$, the second equation reduces to $C_3 = 0$. The trivial solution is obtained, so zero is not an eigenvalue.

Determination of Negative Eigenvalues: $\lambda = -\gamma^2$

Suppose that λ is negative. Then equation (1) becomes

$$\frac{d^2\phi}{dx^2} = -\gamma^2\phi.$$

Its solution can be written in terms of sine and cosine.

$$\phi(x) = C_5\cos\gamma x + C_6\sin\gamma x$$

Apply the boundary conditions to determine C_5 and C_6.

$$\begin{aligned}\phi(0) &= C_5 = 0\\ \phi(\pi) &= C_5\cos\gamma\pi + C_6\sin\gamma\pi = 0\end{aligned}$$

Since $C_5 = 0$, the second equation reduces to $C_6 \sin\gamma\pi = 0$. To avoid getting the trivial solution, we insist that $C_6 \neq 0$. Then

$$\begin{aligned}\sin\gamma\pi &= 0\\ \gamma\pi &= n\pi, \quad n = 1, 2, \ldots\\ \gamma_n &= n, \quad n = 1, 2, \ldots.\end{aligned}$$

The eigenfunctions associated with these eigenvalues for λ are

$$\phi(x) = C_6\sin\gamma x \quad \rightarrow \quad \phi_n(x) = \sin nx, \quad n = 1, 2, \ldots.$$

Method 1 - Using Term-by-Term Differentiation

The eigenfunctions of the Helmholtz equation form a complete set, so the unknown function u can be expanded in terms of them.

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin nx$$

To determine the generalized Fourier coefficients $a_n(t)$, substitute this expansion into the PDE.

$$\begin{aligned}u_{tt} &= c^2u_{xx} + e^t\sin 5x\\ \frac{\partial^2}{\partial t^2}\sum_{n=1}^{\infty} a_n(t)\sin nx &= c^2\frac{\partial^2}{\partial x^2}\sum_{n=1}^{\infty} a_n(t)\sin nx + e^t\sin 5x\end{aligned}$$

Because u satisfies homogeneous boundary conditions and u, $\partial u/\partial x$, and $\partial u/\partial t$ are continuous (reasonable assumptions for the displacement of a homogeneous elastic string), the two series can in fact be differentiated term by term.

$$\sum_{n=1}^{\infty} \frac{d^2 a_n}{dt^2} \sin nx = c^2 \sum_{n=1}^{\infty} a_n(t) \frac{d^2}{dx^2} \sin nx + e^t \sin 5x$$

The operator applied to the eigenfunction is equal to the eigenvalue times the eigenfunction.

$$\sum_{n=1}^{\infty} \frac{d^2 a_n}{dt^2} \sin nx = c^2 \sum_{n=1}^{\infty} a_n(t) \lambda_n \sin nx + e^t \sin 5x$$

Bring both series to the left side and combine them.

$$\sum_{n=1}^{\infty} \left[\frac{d^2 a_n}{dt^2} - c^2 \lambda_n a_n(t) \right] \sin nx = e^t \sin 5x$$

Both sides are essentially Fourier sine series expansions. Matching the time-dependent coefficients, we obtain the following ODEs.

$$\begin{cases} \dfrac{d^2 a_n}{dt^2} - c^2 \lambda_n a_n = 0 & n \neq 5 \\ \dfrac{d^2 a_5}{dt^2} - c^2 \lambda_5 a_5 = e^t & n = 5 \end{cases}$$

Since $\lambda_n = -n^2$, the first equation becomes

$$\frac{d^2 a_n}{dt^2} + c^2 n^2 a_n = 0.$$

Its general solution can be written in terms of sine and cosine.

$$a_n(t) = C_7 \cos cnt + C_8 \sin cnt, \quad n \neq 5$$

The second equation becomes

$$\frac{d^2 a_5}{dt^2} + c^2 5^2 a_5 = e^t.$$

Because it is linear, the general solution is the sum of a complementary solution and a particular solution.

$$a_5 = a_c + a_p$$

The complementary solution satisfies the associated homogeneous equation.

$$\frac{d^2 a_c}{dt^2} + c^2 5^2 a_c = 0$$

It is

$$a_c(t) = C_9 \cos 5ct + C_{10} \sin 5ct.$$

The inhomogeneous term is an exponential, so the particular solution has the form $a_p = C_0 e^t$. Substitute it into the ODE to determine C_0.

$$\frac{d^2 a_p}{dt^2} + c^2 5^2 a_p = e^t \quad \rightarrow \quad C_0 e^t + c^2 5^2 C_0 e^t = e^t$$

$$C_0 + 25c^2 C_0 = 1$$

$$C_0 = \frac{1}{1 + 25c^2}$$

Hence, the general solution for a_5 is

$$a_5(t) = C_9 \cos 5ct + C_{10} \sin 5ct + \frac{e^t}{1+25c^2}.$$

Use the initial conditions for u in combination with the eigenfunction expansion to determine those for a_5 and a_n.

$$u(x,0) = \sum_{n=1}^{\infty} a_n(0) \sin nx = 0$$
$$u_t(x,0) = \sum_{n=1}^{\infty} \frac{da_n}{dt}(0) \sin nx = \sin 3x$$

The first equation implies that $a_n(0) = 0$ for all n. The second equation implies that

$$\begin{cases} \frac{da_n}{dt}(0) = 0 & n \neq 3 \\ \frac{da_n}{dt}(0) = 1 & n = 3 \end{cases}.$$

Since a_3 has a different initial condition from the rest, we will treat it separately and use different constants for it.

$$a_3(t) = C_{11} \cos 3ct + C_{12} \sin 3ct$$

Apply $a_n(0) = 0$ for all n.

$$a_n(0) = C_7 = 0, \quad n \neq 3,\ n \neq 5 \qquad a_5(0) = C_9 + \frac{1}{1+25c^2} = 0 \qquad a_3(0) = C_{11} = 0$$
$$C_9 = -\frac{1}{1+25c^2}$$

Now apply the second initial condition for each.

$$\frac{da_n}{dt}(0) = cn(C_8) = 0, \quad n \neq 3,\ n \neq 5 \qquad \frac{da_5}{dt}(0) = 5c(C_{10}) + \frac{1}{1+25c^2} = 0 \qquad \frac{da_3}{dt}(0) = 3c(C_{12}) = 1$$
$$C_8 = 0 \qquad C_{10} = -\frac{1}{5c(1+25c^2)} \qquad C_{12} = \frac{1}{3c}$$

Thus,

$$\begin{cases} a_n(t) = 0 & n \neq 3,\ n \neq 5 \\ a_5(t) = -\frac{1}{1+25c^2} \cos 5ct - \frac{1}{5c(1+25c^2)} \sin 5ct + \frac{e^t}{1+25c^2} & n = 5 \\ a_3(t) = \frac{1}{3c} \sin 3ct & n = 3 \end{cases}.$$

So then

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin nx = \frac{1}{3c} \sin 3ct \sin 3x + \left[-\frac{1}{1+25c^2} \cos 5ct - \frac{1}{5c(1+25c^2)} \sin 5ct + \frac{e^t}{1+25c^2} \right] \sin 5x.$$

Therefore,

$$u(x,t) = \frac{1}{3c} \sin 3ct \sin 3x + \frac{1}{1+25c^2} \left(e^t - \cos 5ct - \frac{1}{5c} \sin 5ct \right) \sin 5x.$$

Method 2 - Without Using Term-by-Term Differentiation

The eigenfunctions of the Helmholtz equation are known to form a complete set, so all of the functions in the PDE can be expanded in terms of them.

$$u(x,t) = \sum_{n=1}^{\infty} A_n(t)\phi_n(x) \quad \to \quad u\phi_m = \sum_{n=1}^{\infty} A_n\phi_n\phi_m \quad \to \quad \int_0^\pi u\phi_n\,dx = A_n \int_0^\pi \phi_n^2\,dx = A_n \cdot \frac{\pi}{2}$$

$$\frac{\partial^2 u}{\partial t^2} = \sum_{n=1}^{\infty} B_n(t)\phi_n(x) \quad \to \quad \frac{\partial^2 u}{\partial t^2}\phi_m = \sum_{n=1}^{\infty} B_n\phi_n\phi_m \quad \to \quad \int_0^\pi \frac{\partial^2 u}{\partial t^2}\phi_n\,dx = B_n \int_0^\pi \phi_n^2\,dx = B_n \cdot \frac{\pi}{2}$$

$$e^t \sin 5x = \sum_{n=1}^{\infty} D_n(t)\phi_n(x) \quad \to \quad e^t \sin 5x\phi_m = \sum_{n=1}^{\infty} D_n\phi_n\phi_m \quad \to \quad e^t \int_0^\pi \sin 5x\phi_n\,dx = D_n \int_0^\pi \phi_n^2\,dx = D_n \cdot \frac{\pi}{2}$$

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} E_n(t)\phi_n(x) \quad \to \quad \frac{\partial^2 u}{\partial x^2}\phi_m = \sum_{n=1}^{\infty} E_n\phi_n\phi_m \quad \to \quad \int_0^\pi \frac{\partial^2 u}{\partial x^2}\phi_n\,dx = E_n \int_0^\pi \phi_n^2\,dx = E_n \cdot \frac{\pi}{2}$$

It should be emphasized that these are generalized Fourier series expansions for the functions, not product solutions that come about from using the method of separation of variables. Solve the latter equations for the generalized Fourier coefficients.

$$A_n(t) = \frac{2}{\pi}\int_0^\pi u\phi_n\,dx$$

$$B_n(t) = \frac{2}{\pi}\int_0^\pi \frac{\partial^2 u}{\partial t^2}\phi_n\,dx = \frac{d^2}{dt^2}\left(\frac{2}{\pi}\int_0^\pi u\phi_n\,dx\right) = \frac{d^2 A_n}{dt^2}$$

$$D_n(t) = \frac{2e^t}{\pi}\int_0^\pi \sin 5x\phi_n\,dx = \begin{cases} 0 & n \neq 5 \\ e^t & n = 5 \end{cases}$$

$$E_n(t) = \frac{2}{\pi}\int_0^\pi \frac{\partial^2 u}{\partial x^2}\phi_n\,dx = \frac{2}{\pi}\left(\underbrace{\frac{\partial u}{\partial x}\phi_n\bigg|_0^\pi}_{=\,0} - \int_0^\pi \frac{\partial u}{\partial x}\frac{d\phi_n}{dx}\,dx\right) = -\frac{2n}{\pi}\int_0^\pi \frac{\partial u}{\partial x}\cos nx\,dx$$

Apply integration by parts once more in order to write E_n in terms of A_n.

$$\begin{aligned} E_n(t) &= -\frac{2n}{\pi}\left[\underbrace{u\cos nx\bigg|_0^\pi}_{=\,0} - \int_0^\pi u(-n\sin nx)\,dx\right] \\ &= -n^2\left(\frac{2}{\pi}\int_0^\pi u\sin nx\,dx\right) \\ &= -n^2 A_n \end{aligned}$$

Now that the coefficients are known, substitute the eigenfunction expansions into the PDE.

$$\begin{gathered} u_{tt} = c^2 u_{xx} + e^t \sin 5x \\ \sum_{n=1}^{\infty} B_n(t)\phi_n(x) = c^2 \sum_{n=1}^{\infty} E_n(t)\phi_n(x) + \sum_{n=1}^{\infty} D_n(t)\phi_n(x) \\ \sum_{n=1}^{\infty} B_n(t)\phi_n(x) = \sum_{n=1}^{\infty} [c^2 E_n(t) + D_n(t)]\phi_n(x) \end{gathered}$$

Thus,

$$B_n(t) = c^2 E_n(t) + D_n(t).$$

Substitute the formulas for B_n, E_n, and D_n to obtain an ODE for A_n exclusively.

$$\begin{cases} \dfrac{d^2 A_n}{dt^2} = -c^2 n^2 A_n & n \neq 5 \\ \dfrac{d^2 A_5}{dt^2} = -c^2 5^2 A_5 + e^t & n = 5 \end{cases}$$

Bring the terms with A_n to the left.

$$\begin{cases} \dfrac{d^2 A_n}{dt^2} + c^2 n^2 A_n = 0 & n \neq 5 \\ \dfrac{d^2 A_5}{dt^2} + c^2 5^2 A_5 = e^t & n = 5 \end{cases}$$

These are the same ODEs that were obtained for a_n in Method 1, since $\lambda_n = -n^2$. The initial conditions are also the same as before, so $A_n(t) = a_n(t)$.

$$\begin{cases} A_n(t) = 0 & n \neq 3,\ n \neq 5 \\ A_5(t) = -\dfrac{1}{1+25c^2}\cos 5ct - \dfrac{1}{5c(1+25c^2)}\sin 5ct + \dfrac{e^t}{1+25c^2} & n = 5 \\ A_3(t) = \dfrac{1}{3c}\sin 3ct & n = 3 \end{cases}.$$

So then

$$u(x,t) = \sum_{n=1}^{\infty} A_n(t)\sin nx = \frac{1}{3c}\sin 3ct \sin 3x + \left[-\frac{1}{1+25c^2}\cos 5ct - \frac{1}{5c(1+25c^2)}\sin 5ct + \frac{e^t}{1+25c^2}\right]\sin 5x.$$

Therefore,

$$u(x,t) = \frac{1}{3c}\sin 3ct \sin 3x + \frac{1}{1+25c^2}\left(e^t - \cos 5ct - \frac{1}{5c}\sin 5ct\right)\sin 5x.$$