Exercise 6

Solve $u_{tt} = c^2 u_{xx} + g(x)\sin\omega t$ for $0 < x < l$, with $u = 0$ at both ends and $u = u_t = 0$ when $t = 0$. For which values of ω can resonance occur? (Resonance means growth in time.)

Solution

Since the PDE is linear and inhomogeneous, we choose to apply the method of eigenfunction expansion to solve it. Consider the eigenvalue problem of the differential operator involving the spatial variable x

$$\frac{d^2}{dx^2}\phi = \lambda\phi \tag{1}$$

with the same boundary conditions as u.

$$\phi(0) = 0$$
$$\phi(l) = 0$$

Values of λ for which the boundary conditions are satisfied are known as the eigenvalues, and the nontrivial solutions associated with them are called the eigenfunctions. Equation (1) is known as the one-dimensional Helmholtz equation; the eigenfunctions for it are known to be orthogonal and form a complete set, which will prove useful later.

Determination of Positive Eigenvalues: $\lambda = \mu^2$

Suppose that λ is positive. Then equation (1) becomes

$$\frac{d^2\phi}{dx^2} = \mu^2\phi.$$

Its solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$\phi(x) = C_1\cosh\mu x + C_2\sinh\mu x$$

Apply the boundary conditions to determine C_1 and C_2.

$$\phi(0) = C_1 = 0$$
$$\phi(l) = C_1\cosh\mu l + C_2\sinh\mu l = 0$$

Since $C_1 = 0$, the second equation reduces to $C_2\sinh\mu l = 0$. Hyperbolic sine is not oscillatory, so the only way this equation is satisfied is if $C_2 = 0$. The trivial solution is obtained, so there are no positive eigenvalues.

Determination of the Zero Eigenvalue: $\lambda = 0$

Suppose that λ is zero. Then equation (1) becomes

$$\frac{d^2\phi}{dx^2} = 0.$$

The general solution is obtained by integrating both sides with respect to x twice.

$$\phi(x) = C_3 x + C_4$$

Apply the boundary conditions to determine C_3 and C_4.

$$\phi(0) = C_4 = 0$$
$$\phi(l) = C_3 l + C_4 = 0$$

Since $C_4 = 0$, the second equation reduces to $C_3 = 0$. The trivial solution is obtained, so zero is not an eigenvalue.

Determination of Negative Eigenvalues: $\lambda = -\gamma^2$

Suppose that λ is negative. Then equation (1) becomes

$$\frac{d^2\phi}{dx^2} = -\gamma^2\phi.$$

Its solution can be written in terms of sine and cosine.

$$\phi(x) = C_5 \cos\gamma x + C_6 \sin\gamma x$$

Apply the boundary conditions to determine C_5 and C_6.

$$\phi(0) = C_5 = 0$$
$$\phi(l) = C_5 \cos\gamma l + C_6 \sin\gamma l = 0$$

Since $C_5 = 0$, the second equation reduces to $C_6 \sin\gamma l = 0$. To avoid getting the trivial solution, we insist that $C_6 \neq 0$. Then

$$\begin{aligned} \sin\gamma l &= 0 \\ \gamma l &= n\pi, \quad n = 1, 2, \ldots \\ \gamma_n &= \frac{n\pi}{l}, \quad n = 1, 2, \ldots. \end{aligned}$$

The eigenfunctions associated with these eigenvalues for λ are

$$\phi(x) = C_6 \sin\gamma x \quad \rightarrow \quad \phi_n(x) = \sin\frac{n\pi x}{l}, \quad n = 1, 2, \ldots.$$

Method 1 - Using Term-by-Term Differentiation

The eigenfunctions of the Helmholtz equation form a complete set, so the unknown function u can be expanded in terms of them.

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\frac{n\pi x}{l}$$

To determine the generalized Fourier coefficients $a_n(t)$, substitute this expansion into the PDE.

$$u_{tt} = c^2 u_{xx} + g(x)\sin\omega t$$
$$\frac{\partial^2}{\partial t^2}\sum_{n=1}^{\infty} a_n(t)\sin\frac{n\pi x}{l} = c^2\frac{\partial^2}{\partial x^2}\sum_{n=1}^{\infty} a_n(t)\sin\frac{n\pi x}{l} + g(x)\sin\omega t$$

Because u satisfies homogeneous boundary conditions and u, $\partial u/\partial x$, and $\partial u/\partial t$ are continuous (reasonable assumptions for the displacement of a homogeneous elastic string), the two series can in fact be differentiated term by term.

$$\sum_{n=1}^{\infty} \frac{d^2 a_n}{dt^2} \sin\frac{n\pi x}{l} = c^2 \sum_{n=1}^{\infty} a_n(t) \frac{d^2}{dx^2} \sin\frac{n\pi x}{l} + g(x)\sin\omega t$$

The operator applied to the eigenfunction is equal to the eigenvalue times the eigenfunction.

$$\sum_{n=1}^{\infty} \frac{d^2 a_n}{dt^2} \sin\frac{n\pi x}{l} = c^2 \sum_{n=1}^{\infty} a_n(t)\lambda_n \sin\frac{n\pi x}{l} + g(x)\sin\omega t$$

Bring both series to the left side and combine them.

$$\sum_{n=1}^{\infty} \left[\frac{d^2 a_n}{dt^2} - c^2\lambda_n a_n(t)\right] \sin\frac{n\pi x}{l} = g(x)\sin\omega t$$

The left side is essentially a Fourier sine series expansion of $g(x)\sin\omega t$. To solve for the term in square brackets, multiply both sides by $\sin(m\pi x/l)$, where m is an integer,

$$\sum_{n=1}^{\infty} \left[\frac{d^2 a_n}{dt^2} - c^2\lambda_n a_n(t)\right] \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l} = g(x)\sin\frac{m\pi x}{l}\sin\omega t$$

and then integrate both sides with respect to x from 0 to l.

$$\int_0^l \sum_{n=1}^{\infty} \left[\frac{d^2 a_n}{dt^2} - c^2\lambda_n a_n(t)\right] \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx = \int_0^l g(x)\sin\frac{m\pi x}{l}\sin\omega t\,dx$$

$g(x)$ is assumed not to be orthogonal to $\sin(m\pi x/l)$ so that the integral on the right side is nonzero. Bring the functions of t in front of the integrals.

$$\sum_{n=1}^{\infty} \left[\frac{d^2 a_n}{dt^2} - c^2\lambda_n a_n(t)\right] \int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx = \sin\omega t\int_0^l g(x)\sin\frac{m\pi x}{l}\,dx$$

Since the eigenfunctions are orthogonal, the integral on the left side is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for one: $n = m$.

$$\left[\frac{d^2 a_n}{dt^2} - c^2\lambda_n a_n(t)\right] \int_0^l \sin^2\frac{n\pi x}{l}\,dx = \sin\omega t\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx$$

Evaluate the integral on the left side.

$$\left[\frac{d^2 a_n}{dt^2} - c^2\lambda_n a_n(t)\right] \cdot \frac{l}{2} = \sin\omega t\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx$$

Multiply both sides by $2/l$ and replace λ_n with $-(n\pi/l)^2$.

$$\frac{d^2 a_n}{dt^2} + c^2\frac{n^2\pi^2}{l^2}a_n = \left[\frac{2}{l}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx\right]\sin\omega t$$

With the help of the method of eigenfunction expansion, the PDE has been reduced to a second-order inhomogeneous ODE. Because the ODE is linear, the general solution is the sum of a complementary solution and a particular solution.

$$a_n = a_c + a_p$$

The complementary solution satisfies the associated homogeneous equation.

$$\frac{d^2 a_c}{dt^2} + c^2 \frac{n^2\pi^2}{l^2} a_c = 0$$

Its general solution can be written in terms of sine and cosine.

$$a_c(t) = C_7 \cos \frac{cn\pi t}{l} + C_8 \sin \frac{cn\pi t}{l}$$

Since the inhomogeneous term is sine and there are no odd derivatives, the particular solution is of the form $a_p = C_0 \sin \omega t$. Substitute it into the ODE to find C_0.

$$\begin{aligned}
\frac{d^2 a_p}{dt^2} + c^2 \frac{n^2\pi^2}{l^2} a_p &= \left[\frac{2}{l}\int_0^l g(x) \sin \frac{n\pi x}{l}\,dx\right] \sin \omega t \\
-\omega^2 C_0 \sin \omega t + c^2 \frac{n^2\pi^2}{l^2} C_0 \sin \omega t &= \left[\frac{2}{l}\int_0^l g(x) \sin \frac{n\pi x}{l}\,dx\right] \sin \omega t \\
c^2 \frac{n^2\pi^2}{l^2} C_0 - \omega^2 C_0 &= \frac{2}{l}\int_0^l g(x) \sin \frac{n\pi x}{l}\,dx \\
C_0 \frac{c^2 n^2 \pi^2 - l^2\omega^2}{l^2} &= \frac{2}{l}\int_0^l g(x) \sin \frac{n\pi x}{l}\,dx \\
C_0 &= \frac{2l}{c^2 n^2\pi^2 - l^2\omega^2}\int_0^l g(x) \sin \frac{n\pi x}{l}\,dx
\end{aligned}$$

Hence, the general solution for a_n is

$$a_n(t) = C_7 \cos \frac{cn\pi t}{l} + C_8 \sin \frac{cn\pi t}{l} + \frac{2l \sin \omega t}{c^2 n^2\pi^2 - l^2\omega^2}\int_0^l g(x) \sin \frac{n\pi x}{l}\,dx.$$

Use the initial conditions for u in combination with the eigenfunction expansion to determine those for a_n.

$$\begin{aligned}
u(x,0) &= \sum_{n=1}^{\infty} a_n(0) \sin \frac{n\pi x}{l} = 0 &\quad\Rightarrow\quad& a_n(0) = 0 \\
u_t(x,0) &= \sum_{n=1}^{\infty} \frac{da_n}{dt}(0) \sin \frac{n\pi x}{l} = 0 &\quad\Rightarrow\quad& \frac{da_n}{dt}(0) = 0
\end{aligned}$$

Apply them both to obtain a system of equations for C_7 and C_8.

$$\begin{aligned}
a(0) &= C_7 = 0 \\
\frac{da_n}{dt}(0) &= \frac{cn\pi}{l}(C_8) + \frac{2l\omega}{c^2 n^2\pi^2 - l^2\omega^2}\int_0^l g(x) \sin \frac{n\pi x}{l}\,dx = 0
\end{aligned}$$

Solving the second equation for C_8 gives

$$C_8 = \left(-\frac{\omega l}{cn\pi}\right)\frac{2l}{c^2n^2\pi^2 - l^2\omega^2}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx.$$

So then

$$\begin{aligned}
a_n(t) &= \left[\left(-\frac{\omega l}{cn\pi}\right)\frac{2l}{c^2n^2\pi^2 - l^2\omega^2}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx\right]\sin\frac{cn\pi t}{l} + \frac{2l\sin\omega t}{c^2n^2\pi^2 - l^2\omega^2}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx \\
&= \frac{2l}{c^2n^2\pi^2 - l^2\omega^2}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx\left(\sin\omega t - \frac{\omega l}{cn\pi}\sin\frac{cn\pi t}{l}\right).
\end{aligned}$$

Therefore,

$$u(x,t) = \sum_{n=1}^{\infty}\frac{2l}{c^2n^2\pi^2 - l^2\omega^2}\int_0^l g(s)\sin\frac{n\pi s}{l}\,ds\left(\sin\omega t - \frac{\omega l}{cn\pi}\sin\frac{cn\pi t}{l}\right)\sin\frac{n\pi x}{l}.$$

The dummy integration variable has been changed to s to distinguish it from x. Resonance occurs when the solution blows up: $c^2n^2\pi^2 - l^2\omega^2 = 0$ (that is, if ω is a positive integer multiple of $c\pi/l$).

Method 2 - Without Using Term-by-Term Differentiation

The eigenfunctions of the Helmholtz equation are known to form a complete set, so all of the functions in the PDE can be expanded in terms of them.

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} A_n(t)\phi_n(x) \quad\rightarrow\quad u\phi_m = \sum_{n=1}^{\infty} A_n\phi_n\phi_m \quad\rightarrow\quad \int_0^l u\phi_n\,dx = A_n\int_0^l \phi_n^2\,dx = A_n\cdot\frac{l}{2} \\
\frac{\partial^2 u}{\partial t^2} &= \sum_{n=1}^{\infty} B_n(t)\phi_n(x) \quad\rightarrow\quad \frac{\partial^2 u}{\partial t^2}\phi_m = \sum_{n=1}^{\infty} B_n\phi_n\phi_m \quad\rightarrow\quad \int_0^l \frac{\partial^2 u}{\partial t^2}\phi_n\,dx = B_n\int_0^l \phi_n^2\,dx = B_n\cdot\frac{l}{2} \\
g(x)\sin\omega t &= \sum_{n=1}^{\infty} D_n(t)\phi_n(x) \quad\rightarrow\quad g(x)\sin\omega t\phi_m = \sum_{n=1}^{\infty} D_n\phi_n\phi_m \quad\rightarrow\quad \sin\omega t\int_0^l g\phi_n\,dx = D_n\int_0^l \phi_n^2\,dx = D_n\cdot\frac{l}{2} \\
\frac{\partial^2 u}{\partial x^2} &= \sum_{n=1}^{\infty} E_n(t)\phi_n(x) \quad\rightarrow\quad \frac{\partial^2 u}{\partial x^2}\phi_m = \sum_{n=1}^{\infty} E_n\phi_n\phi_m \quad\rightarrow\quad \int_0^l \frac{\partial^2 u}{\partial x^2}\phi_n\,dx = E_n\int_0^l \phi_n^2\,dx = E_n\cdot\frac{l}{2}
\end{aligned}$$

It should be emphasized that these are generalized Fourier series expansions for the functions, not product solutions that come about from using the method of separation of variables. Solve the latter equations for the generalized Fourier coefficients.

$$\begin{aligned}
A_n(t) &= \frac{2}{l}\int_0^l u\phi_n\,dx \\
B_n(t) &= \frac{2}{l}\int_0^l \frac{\partial^2 u}{\partial t^2}\phi_n\,dx = \frac{d^2}{dt^2}\left(\frac{2}{l}\int_0^l u\phi_n\,dx\right) = \frac{d^2 A_n}{dt^2} \\
D_n(t) &= \frac{2\sin\omega t}{l}\int_0^l g(x)\phi_n\,dx \\
E_n(t) &= \frac{2}{l}\int_0^l \frac{\partial^2 u}{\partial x^2}\phi_n\,dx = \frac{2}{l}\left(\underbrace{\frac{\partial u}{\partial x}\phi_n\bigg|_0^l}_{=0} - \int_0^l \frac{\partial u}{\partial x}\frac{d\phi_n}{dx}\,dx\right) = -\frac{2n\pi}{l^2}\int_0^l \frac{\partial u}{\partial x}\cos\frac{n\pi x}{l}\,dx
\end{aligned}$$

Apply integration by parts once more in order to write E_n in terms of A_n.

$$\begin{aligned} E_n(t) &= -\frac{2n\pi}{l^2}\left[\underbrace{u\cos\frac{n\pi x}{l}\bigg|_0^l}_{=0} - \int_0^l u\left(-\frac{n\pi}{l}\sin\frac{n\pi x}{l}\right)dx\right] \\ &= -\frac{n^2\pi^2}{l^2}\left(\frac{2}{l}\int_0^l u\sin\frac{n\pi x}{l}\,dx\right) \\ &= -\frac{n^2\pi^2}{l^2}A_n \end{aligned}$$

Now that the coefficients are known, substitute the eigenfunction expansions into the PDE.

$$\begin{aligned} u_{tt} &= c^2u_{xx} + g(x)\sin\omega t \\ \sum_{n=1}^\infty B_n(t)\phi_n(x) &= c^2\sum_{n=1}^\infty E_n(t)\phi_n(x) + \sum_{n=1}^\infty D_n(t)\phi_n(x) \\ \sum_{n=1}^\infty B_n(t)\phi_n(x) &= \sum_{n=1}^\infty [c^2E_n(t) + D_n(t)]\phi_n(x) \end{aligned}$$

Thus,

$$B_n(t) = c^2E_n(t) + D_n(t).$$

Substitute the formulas for B_n, E_n, and D_n to obtain an ODE for A_n exclusively.

$$\frac{d^2A_n}{dt^2} = -c^2\frac{n^2\pi^2}{l^2}A_n + \frac{2\sin\omega t}{l}\int_0^l g(x)\phi_n\,dx$$

Bring the term with A_n to the left side and replace ϕ_n with $\sin(n\pi x/l)$.

$$\frac{d^2A_n}{dt^2} + c^2\frac{n^2\pi^2}{l^2}A_n = \left[\frac{2}{l}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx\right]\sin\omega t$$

This is the same ODE that was obtained for a_n in Method 1. The initial conditions are also the same as before, so $A_n(t) = a_n(t)$.

$$A_n(t) = \frac{2l}{c^2n^2\pi^2 - l^2\omega^2}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx\left(\sin\omega t - \frac{\omega l}{cn\pi}\sin\frac{cn\pi t}{l}\right)$$

Therefore,

$$u(x,t) = \sum_{n=1}^\infty \frac{2l}{c^2n^2\pi^2 - l^2\omega^2}\int_0^l g(s)\sin\frac{n\pi s}{l}\,ds\left(\sin\omega t - \frac{\omega l}{cn\pi}\sin\frac{cn\pi t}{l}\right)\sin\frac{n\pi x}{l}.$$

The dummy integration variable has been changed to s to distinguish it from x. Resonance occurs when the solution blows up: $c^2n^2\pi^2 - l^2\omega^2 = 0$ (that is, if ω is a positive integer multiple of $c\pi/l$).