# Exercise 8

Solve $u_t = ku_{xx}$ in $(0, l)$, with $u(0, t) = 0$, $u(l, t) = At$, $u(x, 0) = 0$, where $A$ is a constant.

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**Solution**

The PDE is the heat equation for a homogeneous one-dimensional rod. The temperature at the right end rises linearly in time, whereas the left end remains at 0° indefinitely. Initially the entire rod is at 0°.

**Method 1 - Using Term-by-Term Differentiation**

For this method to work, the boundary conditions must be made homogeneous. Make the change of variables $v(x, t) = u(x, t) - r(x, t)$, where $r(x, t)$ is some function that satisfies the boundary conditions.

$$r(0, t) = 0$$
$$r(l, t) = At$$

A suitable function is

$$r(x, t) = \frac{x}{l}At.$$

As a result, the initial and boundary conditions for $v$ are

$$v(0, t) = u(0, t) - r(0, t) = 0 - 0 = 0$$
$$v(l, t) = u(l, t) - r(l, t) = At - At = 0$$
$$v(x, 0) = u(x, 0) - r(x, 0) = 0 - 0 = 0.$$

Substitute $u(x, t) = v(x, t) + r(x, t)$ into the PDE now to find the one that $v$ satisfies.

$$v_t + r_t = k(v_{xx} + r_{xx})$$

Distribute $k$ and bring $r_t$ to the right side.

$$v_t = kv_{xx} + kr_{xx} - r_t$$

Consequently, the PDE for $v$ is

$$v_t = kv_{xx} - \frac{A}{l}x. \tag{1}$$

Since it is linear and inhomogeneous, we choose to apply the method of eigenfunction expansion to solve it. Consider the eigenvalue problem of the differential operator involving the spatial variable $x$

$$\frac{d^2}{dx^2}\phi = \lambda\phi \tag{2}$$

with the same boundary conditions as $v$.

$$\phi(0) = 0$$
$$\phi(l) = 0$$

Values of $\lambda$ for which the boundary conditions are satisfied are known as the eigenvalues, and the nontrivial solutions associated with them are called the eigenfunctions. Equation (2) is known as the one-dimensional Helmholtz equation; the eigenfunctions for it are known to be orthogonal and form a complete set, which will prove useful later.

**Determination of Positive Eigenvalues: $\lambda = \mu^2$**

Suppose that $\lambda$ is positive. Then equation (2) becomes

$$\frac{d^2\phi}{dx^2} = \mu^2\phi.$$

Its solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$\phi(x) = C_1 \cosh \mu x + C_2 \sinh \mu x$$

Apply the boundary conditions to determine $C_1$ and $C_2$.

$$\begin{aligned}\phi(0) &= C_1 = 0 \\ \phi(l) &= C_1 \cosh \mu l + C_2 \sinh \mu l = 0\end{aligned}$$

Since $C_1 = 0$, the second equation reduces to $C_2 \sinh \mu l = 0$. Hyperbolic sine is not oscillatory, so the only way this equation is satisfied is if $C_2 = 0$. The trivial solution is obtained, so there are no positive eigenvalues.

**Determination of the Zero Eigenvalue: $\lambda = 0$**

Suppose that $\lambda$ is zero. Then equation (2) becomes

$$\frac{d^2\phi}{dx^2} = 0.$$

The general solution is obtained by integrating both sides with respect to $x$ twice.

$$\phi(x) = C_3 x + C_4$$

Apply the boundary conditions to determine $C_3$ and $C_4$.

$$\begin{aligned}\phi(0) &= C_4 = 0 \\ \phi(l) &= C_3 l + C_4 = 0\end{aligned}$$

Since $C_4 = 0$, the second equation reduces to $C_3 = 0$. The trivial solution is obtained, so zero is not an eigenvalue.

**Determination of Negative Eigenvalues: $\lambda = -\gamma^2$**

Suppose that $\lambda$ is negative. Then equation (2) becomes

$$\frac{d^2\phi}{dx^2} = -\gamma^2\phi.$$

Its solution can be written in terms of sine and cosine.

$$\phi(x) = C_5 \cos\gamma x + C_6 \sin\gamma x$$

Apply the boundary conditions to determine $C_5$ and $C_6$.

$$\begin{aligned} \phi(0) &= C_5 = 0 \\ \phi(l) &= C_5 \cos\gamma l + C_6 \sin\gamma l = 0 \end{aligned}$$

Since $C_5 = 0$, the second equation reduces to $C_6 \sin\gamma l = 0$. To avoid getting the trivial solution, we insist that $C_6 \neq 0$. Then

$$\begin{aligned} \sin\gamma l &= 0 \\ \gamma l &= n\pi, \quad n = 1, 2, \ldots \\ \gamma_n &= \frac{n\pi}{l}, \quad n = 1, 2, \ldots. \end{aligned}$$

The eigenfunctions associated with these eigenvalues for $\lambda$ are

$$\phi(x) = C_6 \sin\gamma x \quad \rightarrow \quad \phi_n(x) = \sin\frac{n\pi x}{l}, \quad n = 1, 2, \ldots$$

and form a complete set, so the unknown function $v$ can be expanded in terms of them.

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\frac{n\pi x}{l}$$

To determine $a_n(t)$, the generalized Fourier coefficients, substitute this formula for $v$ into the PDE it satisfies, equation (1).

$$\frac{\partial}{\partial t}\sum_{n=1}^{\infty} a_n(t) \sin\frac{n\pi x}{l} = k\frac{\partial^2}{\partial x^2}\sum_{n=1}^{\infty} a_n(t) \sin\frac{n\pi x}{l} - \frac{A}{l}x$$

Because $v$ satisfies homogeneous boundary conditions and $v$, $\partial v/\partial x$, and $\partial v/\partial t$ are continuous (reasonable assumptions for the temperature profile in a homogeneous solid rod), the two series can in fact be differentiated term by term.

$$\sum_{n=1}^{\infty} \frac{da_n}{dt} \sin\frac{n\pi x}{l} = k\sum_{n=1}^{\infty} a_n(t) \frac{d^2}{dx^2} \sin\frac{n\pi x}{l} - \frac{A}{l}x$$

The operator applied to the eigenfunction is equal to the eigenvalue times the eigenfunction.

$$\sum_{n=1}^{\infty} \frac{da_n}{dt} \sin\frac{n\pi x}{l} = k\sum_{n=1}^{\infty} a_n(t)\lambda_n \sin\frac{n\pi x}{l} - \frac{A}{l}x$$

Bring both series to the left side and combine them.

$$\sum_{n=1}^{\infty}\left[\frac{da_n}{dt}-ka_n(t)\lambda_n\right]\sin\frac{n\pi x}{l}=-\frac{A}{l}x$$

The left side is essentially a Fourier sine series expansion of the function on the right side. To solve for the term enclosed in square brackets, multiply both sides by $\sin m\pi x/l$, where $m$ is an integer,

$$\sum_{n=1}^{\infty}\left[\frac{da_n}{dt}-ka_n(t)\lambda_n\right]\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}=-\frac{A}{l}x\sin\frac{m\pi x}{l}$$

and then integrate both sides with respect to $x$ from 0 to $l$.

$$\int_0^l\sum_{n=1}^{\infty}\left[\frac{da_n}{dt}-ka_n(t)\lambda_n\right]\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx=\int_0^l\frac{-A}{l}x\sin\frac{m\pi x}{l}\,dx$$

Bring the constants in front of the integrals.

$$\sum_{n=1}^{\infty}\left[\frac{da_n}{dt}-ka_n(t)\lambda_n\right]\int_0^l\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx=-\frac{A}{l}\int_0^l x\sin\frac{m\pi x}{l}\,dx$$

Because the sine functions are orthogonal, the integral on the left side is zero if $n\neq m$. As a result, every term in the infinite series vanishes except for one: $n=m$.

$$\left[\frac{da_n}{dt}-ka_n(t)\lambda_n\right]\int_0^l\sin^2\frac{n\pi x}{l}\,dx=-\frac{A}{l}\int_0^l x\sin\frac{n\pi x}{l}\,dx$$

Evaluate the integrals.

$$\left[\frac{da_n}{dt}-ka_n(t)\lambda_n\right]\cdot\frac{l}{2}=-\frac{A}{l}\left[-\frac{(-1)^n l^2}{n\pi}\right]$$

Multiply both sides by $2/l$ and simplify both sides.

$$\frac{da_n}{dt}-k\lambda_n a_n=\frac{2(-1)^n A}{n\pi}\tag{3}$$

With the help of the method of eigenfunction expansion, the PDE for $v$ has been reduced to a first-order inhomogeneous ODE, which can be solved with an integrating factor $I$.

$$I=\exp\left[\int^t(-k\lambda_n)\,ds\right]=e^{-k\lambda_n t}$$

Multiply both sides of the previous equation by $I$.

$$\frac{da_n}{dt}e^{-k\lambda_n t}-k\lambda_n a_n e^{-k\lambda_n t}=\frac{2(-1)^n A}{n\pi}e^{-k\lambda_n t}$$

The left side can be written as $d/dt(Ia_n)$ by the product rule.

$$\frac{d}{dt}(e^{-k\lambda_n t}a_n)=\frac{2(-1)^n A}{n\pi}e^{-k\lambda_n t}$$

Integrate both sides with respect to $t$.

$$e^{-k\lambda_n t}a_n = \int^t \frac{2(-1)^n A}{n\pi} e^{-k\lambda_n s}\, ds + C_7$$

Evaluate the remaining integral.

$$e^{-k\lambda_n t}a_n = \frac{2(-1)^n A}{n\pi}\frac{1}{-k\lambda_n} e^{-k\lambda_n t} + C_7$$

The general solution for $a_n$ is thus

$$a_n(t) = -\frac{2(-1)^n A}{n\pi k\lambda_n} + C_7 e^{k\lambda_n t}. \tag{4}$$

In order to determine $C_7$, use the initial condition for $v$ in combination with the eigenfunction expansion.

$$v(x,0) = \sum_{n=1}^{\infty} a_n(0)\sin\frac{n\pi x}{l} = 0 \quad \rightarrow \quad a_n(0) = 0$$

Now set $t = 0$ in equation (4) and use this initial condition.

$$a_n(0) = -\frac{2(-1)^n A}{n\pi k\lambda_n} + C_7 = 0 \quad \rightarrow \quad C_7 = \frac{2(-1)^n A}{n\pi k\lambda_n}$$

So then

$$\begin{aligned} a_n(t) &= -\frac{2(-1)^n A}{n\pi k\lambda_n} + \frac{2(-1)^n A}{n\pi k\lambda_n} e^{k\lambda_n t} \\ &= -\frac{2(-1)^n A}{n\pi k\lambda_n}(1 - e^{k\lambda_n t}). \end{aligned}$$

Substitute $\lambda_n = -\gamma_n^2 = -(n\pi/l)^2$ here.

$$= \frac{2(-1)^n A l^2}{n^3\pi^3 k}\left[1 - \exp\left(-k\frac{n^2\pi^2}{l^2}t\right)\right]$$

Consequently,

$$v(x,t) = \sum_{n=1}^{\infty} \frac{2(-1)^n A l^2}{n^3\pi^3 k}\left[1 - \exp\left(-k\frac{n^2\pi^2}{l^2}t\right)\right]\sin\frac{n\pi x}{l}$$

and, since $u(x,t) = v(x,t) + r(x,t)$,

$$u(x,t) = \frac{x}{l}At + \sum_{n=1}^{\infty} \frac{2(-1)^n A l^2}{n^3\pi^3 k}\left[1 - \exp\left(-k\frac{n^2\pi^2}{l^2}t\right)\right]\sin\frac{n\pi x}{l}.$$

Therefore,

$$u(x,t) = \frac{x}{l}At + \frac{2Al^2}{\pi^3 k}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}\left[1 - \exp\left(-k\frac{n^2\pi^2}{l^2}t\right)\right]\sin\frac{n\pi x}{l}, \quad 0 \le x < l.$$

Note that this solution does not satisfy $u_t = ku_{xx}$ at $x = l$.

## Method 2 - The Method of Separation of Variables

Because the inhomogeneous term in equation (1) is independent of $t$, the PDE for $v$ can be made homogeneous by making the substitution $v(x,t) = w(x,t) + q(x)$.

$$v_t = kv_{xx} - \frac{A}{l}x \quad \rightarrow \quad w_t = k\left(w_{xx} + \frac{d^2q}{dx^2}\right) - \frac{A}{l}x$$
$$w_t = kw_{xx} + k\frac{d^2q}{dx^2} - \frac{A}{l}x$$

If we set

$$k\frac{d^2q}{dx^2} - \frac{A}{l}x = 0,$$

then the previous equation becomes

$$w_t = kw_{xx}.$$

Solve the ODE for $q$ by integrating both sides with respect to $x$ twice.

$$\frac{d^2q}{dx^2} = \frac{A}{kl}x \quad \rightarrow \quad \frac{dq}{dx} = \frac{A}{2kl}x^2 + C_8 \quad \rightarrow \quad q(x) = \frac{A}{6kl}x^3 + C_8x + C_9$$

Let $q(x)$ have the same boundary conditions as $v(x,t)$: $q(0) = 0$ and $q(l) = 0$. Apply them to determine $C_8$ and $C_9$.

$$q(0) = C_9 = 0$$
$$q(l) = \frac{A}{6k}l^2 + C_8l + C_9 = 0$$

Solving the second equation for $C_8$ yields

$$C_8 = -\frac{A}{6k}l.$$

So then

$$q(x) = \frac{A}{6kl}x^3 - \frac{A}{6k}lx$$
$$= \frac{A}{6kl}x(x^2 - l^2).$$

Before solving the PDE for $w$, determine the initial and boundary conditions associated with it.

$$w(0,t) = v(0,t) - q(0) = 0 - 0 = 0$$
$$w(l,t) = v(l,t) - q(l) = 0 - 0 = 0$$
$$w(x,0) = v(x,0) - q(x) = 0 - q(x) = \frac{A}{6kl}x(l^2 - x^2)$$

The PDE and its boundary conditions are linear and homogeneous, so the method of separation of variables can be applied to solve it. Assume a product solution $w = X(x)T(t)$ and substitute it into the PDE

$$w_t = kw_{xx} \quad \rightarrow \quad XT' = kX''T$$

and the boundary conditions.

$$\begin{array}{ccccc} w(0,t)=0 & \to & X(0)T(t)=0 & \to & X(0)=0 \\ w(l,t)=0 & \to & X(l)T(t)=0 & \to & X(l)=0 \end{array}$$

Now separate variables in the PDE: bring all constants and functions of $t$ to the left side and all functions of $x$ to the right side.

$$\frac{T'}{kT}=\frac{X''}{X}$$

The only way a function of $t$ can be equal to a function of $x$ is if both are equal to a constant $\lambda$.

$$\frac{T'}{kT}=\frac{X''}{X}=\lambda$$

Multiplying both sides of the second equation by $X$,

$$X''=\lambda X,$$

we obtain the same eigenvalue problem we solved earlier for $\phi$. We found that the eigenvalues are $\lambda=\lambda_n=-(n\pi/l)^2$ ($n=1,2,\ldots$), and the eigenfunctions associated with them are

$$X_n(x)=\sin\frac{n\pi x}{l},\quad n=1,2,\ldots.$$

Solve the related ODE for $T$ with this value for $\lambda$.

$$\frac{T'}{kT}=-\frac{n^2\pi^2}{l^2}$$

Multiply both sides by $kT$.

$$T'=-k\frac{n^2\pi^2}{l^2}T$$

The solution can be written in terms of the exponential function.

$$T(t)=C_{10}\exp\left(-k\frac{n^2\pi^2}{l^2}t\right)$$

According to the principle of superposition, the general solution for $w$ is a linear combination of the eigenfunctions over all the eigenvalues.

$$w(x,t)=\sum_{n=1}^{\infty}Z_n\exp\left(-k\frac{n^2\pi^2}{l^2}t\right)\sin\frac{n\pi x}{l}$$

Apply the initial condition now to determine the coefficients $Z_n$.

$$w(x,0)=\sum_{n=1}^{\infty}Z_n\sin\frac{n\pi x}{l}=\frac{A}{6kl}x(l^2-x^2)$$

Multiply both sides by $\sin(m\pi x/l)$

$$\sum_{n=1}^{\infty}Z_n\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}=\frac{A}{6kl}x(l^2-x^2)\sin\frac{m\pi x}{l}$$

and integrate both sides with respect to $x$ from 0 to $l$.

$$\int_0^l \sum_{n=1}^{\infty} Z_n \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l}\,dx = \int_0^l \frac{A}{6kl}x(l^2 - x^2)\sin\frac{m\pi x}{l}\,dx$$

Bring the constants in front of the integrals.

$$\sum_{n=1}^{\infty} Z_n \int_0^l \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l}\,dx = \frac{A}{6kl}\int_0^l x(l^2 - x^2)\sin\frac{m\pi x}{l}\,dx$$

Because the eigenfunctions are orthogonal, the integral on the left side is zero for $n \neq m$. As a result, every term in the infinite series vanishes except for one: $n = m$.

$$Z_n \int_0^l \sin^2\frac{n\pi x}{l}\,dx = \frac{A}{6kl}\int_0^l x(l^2 - x^2)\sin\frac{n\pi x}{l}\,dx$$

Evaluate the integrals.

$$Z_n \cdot \frac{l}{2} = \frac{A}{6kl}\left[-\frac{6(-1)^n l^4}{n^3\pi^3}\right]$$

Solve for $Z_n$.

$$Z_n = -\frac{2(-1)^n A l^2}{kn^3\pi^3}$$

Consequently,

$$\begin{aligned} w(x,t) &= \sum_{n=1}^{\infty}\left[-\frac{2(-1)^n A l^2}{kn^3\pi^3}\right]\exp\left(-k\frac{n^2\pi^2}{l^2}t\right)\sin\frac{n\pi x}{l} \\ &= -\frac{2Al^2}{k\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\exp\left(-k\frac{n^2\pi^2}{l^2}t\right)\sin\frac{n\pi x}{l} \end{aligned}$$

and, since $v(x,t) = w(x,t) + q(x)$,

$$v(x,t) = \frac{A}{6kl}x(x^2 - l^2) - \frac{2Al^2}{k\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\exp\left(-k\frac{n^2\pi^2}{l^2}t\right)\sin\frac{n\pi x}{l}.$$

Therefore, since $u(x,t) = v(x,t) + r(x,t)$,

$$u(x,t) = \frac{x}{l}At + \frac{A}{6kl}x(x^2 - l^2) - \frac{2Al^2}{k\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\exp\left(-k\frac{n^2\pi^2}{l^2}t\right)\sin\frac{n\pi x}{l}, \quad 0 \le x \le l.$$

This solution converges very quickly and does satisfy the PDE at $x = l$.

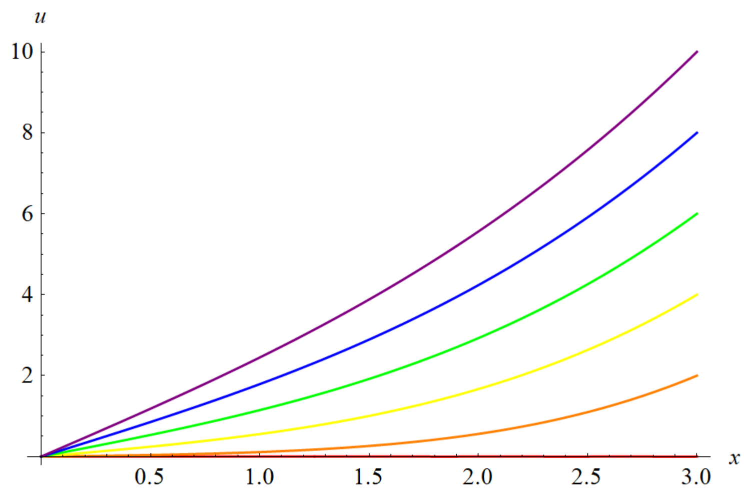


Figure 1: This figure shows $u(x,t)$ at various times for $k = 2$, $l = 3$, and $A = 4$. The curves in red, orange, yellow, green, blue, and purple correspond to $t = 0$, $t = 0.5$, $t = 1$, $t = 1.5$, $t = 2$, and $t = 2.5$, respectively. They are approximate, as only the first 10 terms in the infinite series have been used. This graph is virtually the same as the one for the solution obtained by Method 1.

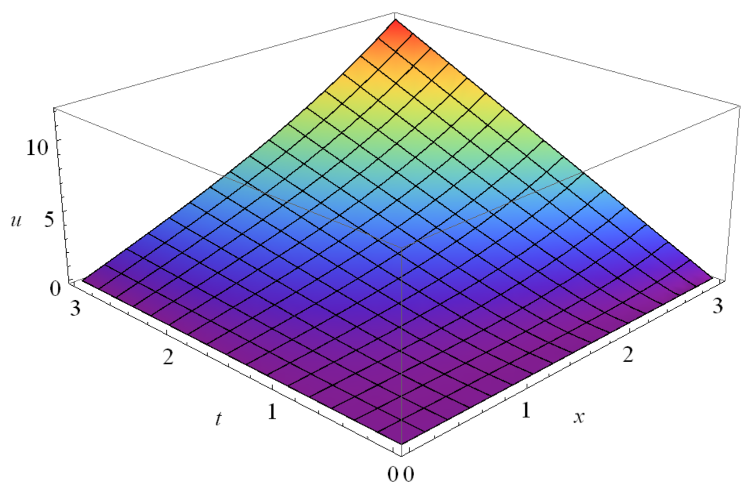


Figure 2: This is a plot of the two-dimensional solution surface $u(x,t)$ in three-dimensional $xyu$-space for $k = 2$, $l = 3$, and $A = 4$. It is approximate, as only the first 10 terms in the infinite series have been used.

## Method 3 - Without Using Term-by-Term Differentiation

Rather than substituting the eigenfunction expansion of $v$ into the PDE and differentiating term by term, we can use an alternative approach to obtain the ODE for the generalized Fourier coefficients $a_n$. Start by expanding the functions in the PDE for $v$ in terms of the eigenfunctions.

$$\begin{aligned}
v(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x) \quad &\to \quad v\phi_m = \sum_{n=1}^{\infty} a_n\phi_n\phi_m \quad &\to \quad \int_0^l v\phi_n\,dx = a_n \int_0^l \phi_n^2\,dx = a_n \cdot \frac{l}{2} \\
\frac{\partial v}{\partial t} = \sum_{n=1}^{\infty} b_n(t)\phi_n(x) \quad &\to \quad \frac{\partial v}{\partial t}\phi_m = \sum_{n=1}^{\infty} b_n\phi_n\phi_m \quad &\to \quad \int_0^l \frac{\partial v}{\partial t}\phi_n\,dx = b_n \int_0^l \phi_n^2\,dx = b_n \cdot \frac{l}{2} \\
\frac{A}{l}x = \sum_{n=1}^{\infty} c_n(t)\phi_n(x) \quad &\to \quad \frac{A}{l}x\phi_m = \sum_{n=1}^{\infty} c_n\phi_n\phi_m \quad &\to \quad \int_0^l \frac{A}{l}x\phi_n\,dx = c_n \int_0^l \phi_n^2\,dx = c_n \cdot \frac{l}{2} \\
\frac{\partial^2 v}{\partial x^2} = \sum_{n=1}^{\infty} d_n(t)\phi_n(x) \quad &\to \quad \frac{\partial^2 v}{\partial x^2}\phi_m = \sum_{n=1}^{\infty} d_n\phi_n\phi_m \quad &\to \quad \int_0^l \frac{\partial^2 v}{\partial x^2}\phi_n\,dx = d_n \int_0^l \phi_n^2\,dx = d_n \cdot \frac{l}{2}
\end{aligned}$$

It should be emphasized that these are generalized Fourier series expansions for the functions, not product solutions that come about from using the method of separation of variables. Solve the latter equations for the generalized Fourier coefficients.

$$\begin{aligned}
a_n(t) &= \frac{2}{l}\int_0^l v\phi_n\,dx \\
b_n(t) &= \frac{2}{l}\int_0^l \frac{\partial v}{\partial t}\phi_n\,dx = \frac{d}{dt}\left(\frac{2}{l}\int_0^l v\phi_n\,dx\right) = \frac{da_n}{dt} \\
c_n(t) &= \frac{2}{l}\int_0^l \frac{A}{l}x\phi_n\,dx = \frac{2A}{l^2}\int_0^l x\sin\frac{n\pi x}{l}\,dx = \frac{2A}{l^2}\left[-\frac{(-1)^n l^2}{n\pi}\right] = -\frac{2(-1)^n A}{n\pi} \\
d_n(t) &= \frac{2}{l}\int_0^l \frac{\partial^2 v}{\partial x^2}\phi_n\,dx = \frac{2}{l}\left(\underbrace{\frac{\partial v}{\partial x}\phi_n\bigg|_0^l}_{=\,0} - \int_0^l \frac{\partial v}{\partial x}\frac{d\phi_n}{dx}\,dx\right) = -\frac{2n\pi}{l^2}\int_0^l \frac{\partial v}{\partial x}\cos\frac{n\pi x}{l}\,dx
\end{aligned}$$

Apply integration by parts once more in order to write $d_n$ in terms of $a_n$ and the boundary conditions for $v$.

$$\begin{aligned}
d_n(t) &= -\frac{2n\pi}{l^2}\left[v(x,t)\cos\frac{n\pi x}{l}\bigg|_0^l - \int_0^l v\left(-\frac{n\pi}{l}\sin\frac{n\pi x}{l}\right)dx\right] \\
&= -\frac{2n\pi}{l^2}\left[\underbrace{v(l,t)\cos n\pi}_{=\,0} - \underbrace{v(0,t)}_{=\,0} + \frac{n\pi}{l}\int_0^l v\sin\frac{n\pi x}{l}\,dx\right] \\
&= -\frac{n^2\pi^2}{l^2}\left(\frac{2}{l}\int_0^l v\sin\frac{n\pi x}{l}\,dx\right) \\
&= -\frac{n^2\pi^2}{l^2}a_n
\end{aligned}$$

Now that the coefficients are known, substitute the eigenfunction expansions into the PDE.

$$\begin{aligned} v_t &= kv_{xx} - \frac{A}{l}x \\ \sum_{n=1}^{\infty} b_n(t)\phi_n(x) &= k\sum_{n=1}^{\infty} d_n(t)\phi_n(x) - \sum_{n=1}^{\infty} c_n(t)\phi_n(x) \\ \sum_{n=1}^{\infty} \frac{da_n}{dt}\phi_n(x) &= k\sum_{n=1}^{\infty}\left(-\frac{n^2\pi^2}{l^2}a_n\right)\phi_n(x) - \sum_{n=1}^{\infty}\left[-\frac{2(-1)^nA}{n\pi}\right]\phi_n(x) \\ \sum_{n=1}^{\infty} \frac{da_n}{dt}\phi_n(x) &= \sum_{n=1}^{\infty}\left[-k\frac{n^2\pi^2}{l^2}a_n + \frac{2(-1)^nA}{n\pi}\right]\phi_n(x) \end{aligned}$$

The summands must be equal as a result.

$$\frac{da_n}{dt}\phi_n(x) = \left[-k\frac{n^2\pi^2}{l^2}a_n + \frac{2(-1)^nA}{n\pi}\right]\phi_n(x)$$

Divide both sides by $\phi_n(x)$ and bring the term with $a_n$ to the left side.

$$\frac{da_n}{dt} + k\frac{n^2\pi^2}{l^2}a_n = \frac{2(-1)^nA}{n\pi}$$

This is the same ODE obtained in equation (3), since $\lambda_n = -(n\pi/l)^2$.

**Method 4 - Mr. Strauss's Way**

The method of eigenfunction expansion will be applied directly to solve for $u$ as Mr. Strauss does in the textbook, that is, without making the boundary conditions homogeneous. The same eigenvalue problem is considered here as before.

$$\frac{d^2}{dx^2}\phi = \lambda\phi, \qquad \begin{aligned} \phi(0) &= 0 \\ \phi(l) &= 0 \end{aligned}$$

It was shown that the eigenvalues are $\lambda_n = -(n\pi/l)^2$ and that the eigenfunctions associated with them are

$$\phi_n(x) = \sin\frac{n\pi x}{l}, \quad n = 1, 2, \ldots.$$

The eigenfunctions of the Helmholtz equation are known to form a complete set, so all of the functions in the PDE can be expanded in terms of them.

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} B_n(t)\phi_n(x) &&\rightarrow \quad u\phi_m = \sum_{n=1}^{\infty} B_n\phi_n\phi_m &&\rightarrow \quad \int_0^l u\phi_n\,dx = B_n\int_0^l \phi_n^2\,dx = B_n\cdot\frac{l}{2} \\
\frac{\partial u}{\partial t} &= \sum_{n=1}^{\infty} D_n(t)\phi_n(x) &&\rightarrow \quad \frac{\partial u}{\partial t}\phi_m = \sum_{n=1}^{\infty} D_n\phi_n\phi_m &&\rightarrow \quad \int_0^l \frac{\partial u}{\partial t}\phi_n\,dx = D_n\int_0^l \phi_n^2(x)\,dx = D_n\cdot\frac{l}{2} \\
\frac{\partial^2 u}{\partial x^2} &= \sum_{n=1}^{\infty} E_n(t)\phi_n(x) &&\rightarrow \quad \frac{\partial^2 u}{\partial x^2}\phi_m = \sum_{n=1}^{\infty} E_n\phi_n\phi_m &&\rightarrow \quad \int_0^l \frac{\partial^2 u}{\partial x^2}\phi_n\,dx = E_n\int_0^l \phi_n^2(x)\,dx = E_n\cdot\frac{l}{2}
\end{aligned}$$

It should be emphasized that these are generalized Fourier series expansions for $u$, $\partial u/\partial t$, and $\partial^2 u/\partial x^2$, not product solutions that come about from using the method of separation of variables. Solve the latter equations for the generalized Fourier coefficients.

$$\begin{aligned}
B_n(t) &= \frac{2}{l}\int_0^l u\phi_n\,dx \\
D_n(t) &= \frac{2}{l}\int_0^l \frac{\partial u}{\partial t}\phi_n\,dx = \frac{d}{dt}\left(\frac{2}{l}\int_0^l u\phi_n\,dx\right) = \frac{dB_n}{dt} \\
E_n(t) &= \frac{2}{l}\int_0^l \frac{\partial^2 u}{\partial x^2}\phi_n\,dx = \frac{2}{l}\Bigg(\underbrace{\frac{\partial u}{\partial x}\phi_n\bigg|_0^l}_{=\,0} - \int_0^l \frac{\partial u}{\partial x}\frac{d\phi_n}{dx}\,dx\Bigg) = -\frac{2n\pi}{l^2}\int_0^l \frac{\partial u}{\partial x}\cos\frac{n\pi x}{l}\,dx
\end{aligned}$$

Apply integration by parts once more in order to write $E_n$ in terms of $B_n$ and the given boundary conditions for $u$.

$$\begin{aligned}
E_n(t) &= -\frac{2n\pi}{l^2}\left[u(x,t)\cos\frac{n\pi x}{l}\bigg|_0^l - \int_0^l u\left(-\frac{n\pi}{l}\sin\frac{n\pi x}{l}\right)dx\right] \\
&= -\frac{2n\pi}{l^2}\left[u(l,t)\cos n\pi - u(0,t) + \frac{n\pi}{l}\int_0^l u\sin\frac{n\pi x}{l}\,dx\right] \\
&= -\frac{2n\pi}{l^2}\left[At\cos n\pi + \frac{n\pi}{2}\left(\frac{2}{l}\int_0^l u\sin\frac{n\pi x}{l}\,dx\right)\right] \\
&= -\frac{2n\pi}{l^2}\left[At(-1)^n + \frac{n\pi}{2}B_n\right] \\
&= -\frac{2n\pi(-1)^n A}{l^2}t - \frac{n^2\pi^2}{l^2}B_n
\end{aligned}$$

Now that the coefficients are known, substitute the eigenfunction expansions into the PDE.

$$u_t = ku_{xx}$$
$$\sum_{n=1}^{\infty} D_n(t)\phi_n(x) = k\sum_{n=1}^{\infty} E_n(t)\phi_n(x)$$
$$\sum_{n=1}^{\infty} \frac{dB_n}{dt}\phi_n(x) = \sum_{n=1}^{\infty} kE_n\phi_n(x)$$

The summands must be equal as a result.

$$\frac{dB_n}{dt}\phi_n(x) = kE_n\phi_n(x)$$

Divide both sides by $\phi_n(x)$ and substitute $E_n(t)$.

$$\frac{dB_n}{dt} = -\frac{2kn\pi(-1)^n A}{l^2}t - k\frac{n^2\pi^2}{l^2}B_n$$

Bring the term with $B_n$ to the left side.

$$\frac{dB_n}{dt} + k\frac{n^2\pi^2}{l^2}B_n = -\frac{2kn\pi(-1)^n A}{l^2}t$$

With the help of the method of eigenfunction expansion, the PDE has been reduced to a first-order inhomogeneous ODE that can be solved with an integrating factor $I$.

$$I = \exp\left(\int^t k\frac{n^2\pi^2}{l^2}\,ds\right) = \exp\left(k\frac{n^2\pi^2}{l^2}t\right)$$

Multiply both sides of the previous equation by $I$.

$$\frac{dB_n}{dt}\exp\left(k\frac{n^2\pi^2}{l^2}t\right) + k\frac{n^2\pi^2}{l^2}B_n\exp\left(k\frac{n^2\pi^2}{l^2}t\right) = -\frac{2kn\pi(-1)^n A}{l^2}t\exp\left(k\frac{n^2\pi^2}{l^2}t\right)$$

The left side can be written as $d/dt(IB_n)$ by the product rule.

$$\frac{d}{dt}\left[\exp\left(k\frac{n^2\pi^2}{l^2}t\right)B_n\right] = -\frac{2kn\pi(-1)^n A}{l^2}t\exp\left(k\frac{n^2\pi^2}{l^2}t\right)$$

Integrate both sides with respect to $t$.

$$\exp\left(k\frac{n^2\pi^2}{l^2}t\right)B_n = \int^t \left[-\frac{2kn\pi(-1)^n A}{l^2}\right]s\exp\left(k\frac{n^2\pi^2}{l^2}s\right)ds + C_{11}$$

Evaluate the integral.

$$\exp\left(k\frac{n^2\pi^2}{l^2}t\right)B_n = \left[-\frac{2kn\pi(-1)^n A}{l^2}\right]\cdot\frac{l^2}{k^2n^4\pi^4}(kn^2\pi^2 t - l^2)\exp\left(k\frac{n^2\pi^2}{l^2}t\right) + C_{11}$$

Divide both sides by $I$ to solve for $B_n$.

$$B_n(t) = \frac{2(-1)^n A}{kn^3\pi^3}(l^2 - kn^2\pi^2 t) + C_{11}\exp\left(-k\frac{n^2\pi^2}{l^2}t\right)$$

In order to determine $C_{11}$, use the initial condition for $u$ in combination with the eigenfunction expansion.

$$u(x,0) = \sum_{n=1}^{\infty} B_n(0)\phi_n(x) = 0 \quad \rightarrow \quad B_n(0) = 0$$

Set $t = 0$ in the previous equation and use the initial condition.

$$B_n(0) = \frac{2(-1)^n A}{kn^3\pi^3}(l^2) + C_{11} = 0 \quad \rightarrow \quad C_{11} = -\frac{2(-1)^n A}{kn^3\pi^3}(l^2)$$

So then

$$\begin{aligned} B_n(t) &= \frac{2(-1)^n A}{kn^3\pi^3}(l^2 - kn^2\pi^2 t) - \frac{2(-1)^n A}{kn^3\pi^3}(l^2)\exp\left(-k\frac{n^2\pi^2}{l^2}t\right) \\ &= \frac{2(-1)^n A}{kn^3\pi^3}\left\{l^2\left[1 - \exp\left(-k\frac{n^2\pi^2}{l^2}t\right)\right] - kn^2\pi^2 t\right\}. \end{aligned}$$

Therefore,

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2(-1)^n A}{kn^3\pi^3}\left\{l^2\left[1 - \exp\left(-k\frac{n^2\pi^2}{l^2}t\right)\right] - kn^2\pi^2 t\right\}\sin\frac{n\pi x}{l}, \quad 0 \le x < l.$$

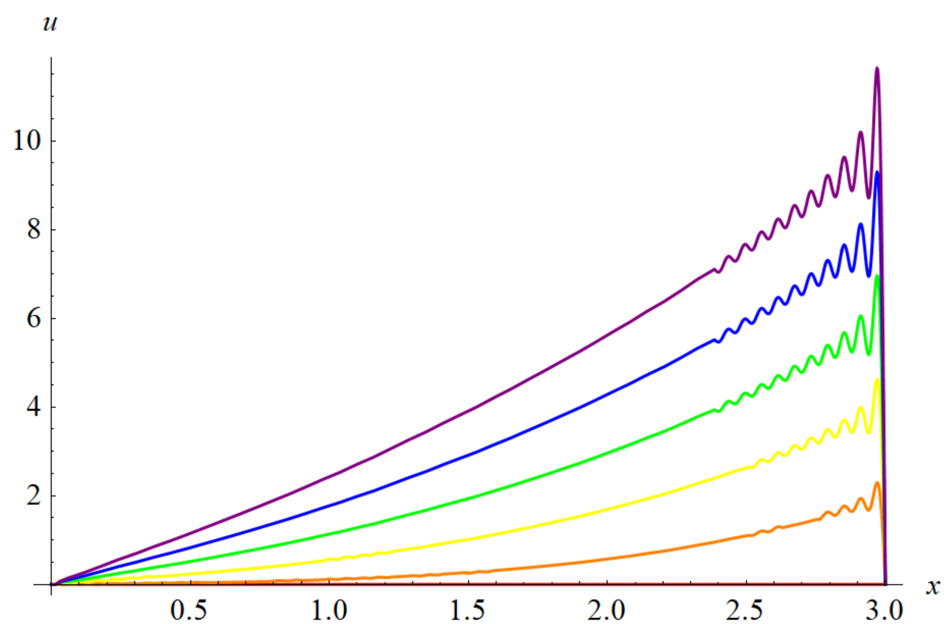


Figure 3: This figure shows $u(x,t)$ at various times for $k = 2$, $l = 3$, and $A = 4$. The curves in red, orange, yellow, green, blue, and purple correspond to $t = 0$, $t = 0.5$, $t = 1$, $t = 1.5$, $t = 2$, and $t = 2.5$, respectively. They are approximate, as only the first 100 terms in the infinite series have been used.

Comparing Figure 3 with Figure 1, we conclude that applying the method of eigenfunction expansion directly as Mr. Strauss does in the textbook results in a series solution that not only converges very slowly but also does not satisfy the inhomogeneous boundary condition at $x = l$. These issues stem from the fact that $\phi(l)$ and $u(l,t)$ are not equal; thus, they can be avoided by making the boundary conditions homogeneous first as was done previously.