

Exercise 12

Carry out the solution of (11) in the case that

$$f(x, t) = F(x) \cos \omega t \quad h(t) = H \cos \omega t \quad k(t) = K \cos \omega t.$$

Solution

Problem (11) with the prescribed functions becomes

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= F(x) \cos \omega t \\ u(0, t) &= H \cos \omega t \quad u(l, t) = K \cos \omega t \\ u(x, 0) &= \phi(x) \quad u_t(x, 0) = \psi(x). \end{aligned} \tag{11}$$

In order to make the boundary conditions homogeneous, start by making the change of variables $v(x, t) = u(x, t) - r(x, t)$, where $r(x, t)$ is some function that satisfies the boundary conditions.

$$\begin{aligned} r(0, t) &= H \cos \omega t \\ r(l, t) &= K \cos \omega t \end{aligned}$$

A suitable function is

$$r(x, t) = \frac{l-x}{l} H \cos \omega t + \frac{x}{l} K \cos \omega t.$$

As a result, the initial and boundary conditions for v are

$$\begin{aligned} v(0, t) &= u(0, t) - r(0, t) = H \cos \omega t - H \cos \omega t = 0 \\ v(l, t) &= u(l, t) - r(l, t) = K \cos \omega t - K \cos \omega t = 0 \\ v(x, 0) &= u(x, 0) - r(x, 0) = \phi(x) - \frac{l-x}{l} H - \frac{x}{l} K \\ v_t(x, 0) &= u_t(x, 0) - r_t(x, 0) = \psi(x). \end{aligned}$$

Substitute $u(x, t) = v(x, t) + r(x, t)$ into the PDE now to find the one that v satisfies.

$$(v_{tt} + r_{tt}) - c^2(v_{xx} + r_{xx}) = F(x) \cos \omega t$$

Distribute c^2 and bring r_{tt} and $c^2 r_{xx}$ to the right side.

$$\begin{aligned} v_{tt} - c^2 v_{xx} &= F(x) \cos \omega t - r_{tt} + \underbrace{c^2 r_{xx}}_{=0} \\ &= F(x) \cos \omega t - \frac{l-x}{l} H \omega^2 \cos \omega t - \frac{x}{l} K \omega^2 \cos \omega t \end{aligned}$$

Consequently, the PDE for v is

$$v_{tt} - c^2 v_{xx} = Q(x, t), \tag{1}$$

where

$$Q(x, t) = \left[F(x) - \frac{l-x}{l} H \omega^2 - \frac{x}{l} K \omega^2 \right] \cos \omega t.$$

Since the PDE is linear and inhomogeneous, we choose to apply the method of eigenfunction expansion to solve it. Consider the eigenvalue problem of the differential operator involving the spatial variable x

$$\frac{d^2}{dx^2} \zeta = \lambda \zeta \tag{2}$$

with the same boundary conditions as v .

$$\begin{aligned}\zeta(0) &= 0 \\ \zeta(l) &= 0\end{aligned}$$

Values of λ for which the boundary conditions are satisfied are known as the eigenvalues, and the nontrivial solutions associated with them are called the eigenfunctions. Equation (2) is known as the one-dimensional Helmholtz equation; the eigenfunctions for it are known to be orthogonal and form a complete set, which will prove useful later.

Determination of Positive Eigenvalues: $\lambda = \mu^2$

Suppose that λ is positive. Then equation (2) becomes

$$\frac{d^2\zeta}{dx^2} = \mu^2\zeta.$$

Its solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$\zeta(x) = C_1 \cosh \mu x + C_2 \sinh \mu x$$

Apply the boundary conditions to determine C_1 and C_2 .

$$\begin{aligned}\zeta(0) &= C_1 = 0 \\ \zeta(l) &= C_1 \cosh \mu l + C_2 \sinh \mu l = 0\end{aligned}$$

Since $C_1 = 0$, the second equation reduces to $C_2 \sinh \mu l = 0$. Hyperbolic sine is not oscillatory, so the only way this equation is satisfied is if $C_2 = 0$. The trivial solution is obtained, so there are no positive eigenvalues.

Determination of the Zero Eigenvalue: $\lambda = 0$

Suppose that λ is zero. Then equation (2) becomes

$$\frac{d^2\zeta}{dx^2} = 0.$$

The general solution is obtained by integrating both sides with respect to x twice.

$$\zeta(x) = C_3x + C_4$$

Apply the boundary conditions to determine C_3 and C_4 .

$$\begin{aligned}\zeta(0) &= C_4 = 0 \\ \zeta(l) &= C_3l + C_4 = 0\end{aligned}$$

Since $C_4 = 0$, the second equation reduces to $C_3 = 0$. The trivial solution is obtained, so zero is not an eigenvalue.

Determination of Negative Eigenvalues: $\lambda = -\gamma^2$

Suppose that λ is negative. Then equation (3) becomes

$$\frac{d^2\zeta}{dx^2} = -\gamma^2\zeta.$$

Its solution can be written in terms of sine and cosine.

$$\zeta(x) = C_5 \cos \gamma x + C_6 \sin \gamma x$$

Apply the boundary conditions to determine C_5 and C_6 .

$$\begin{aligned}\zeta(0) &= C_5 = 0 \\ \zeta(l) &= C_5 \cos \gamma l + C_6 \sin \gamma l = 0\end{aligned}$$

Since $C_5 = 0$, the second equation reduces to $C_6 \sin \gamma l = 0$. To avoid getting the trivial solution, we insist that $C_6 \neq 0$. Then

$$\begin{aligned}\sin \gamma l &= 0 \\ \gamma l &= n\pi, \quad n = 1, 2, \dots \\ \gamma_n &= \frac{n\pi}{l}, \quad n = 1, 2, \dots\end{aligned}$$

The eigenfunctions associated with these eigenvalues for λ are

$$\zeta(x) = C_6 \sin \gamma x \quad \rightarrow \quad \zeta_n(x) = \sin \frac{n\pi x}{l}, \quad n = 1, 2, \dots$$

Method 1 - Using Term-by-Term Differentiation

The eigenfunctions of the Helmholtz equation form a complete set, so the unknown function v can be expanded in terms of them.

$$v(x, t) = \sum_{n=1}^{\infty} a_n(t) \sin \frac{n\pi x}{l}$$

To determine the generalized Fourier coefficients $a_n(t)$, substitute this expansion into the PDE.

$$\begin{aligned}v_{tt} - c^2 v_{xx} &= Q(x, t) \\ \frac{\partial^2}{\partial t^2} \sum_{n=1}^{\infty} a_n(t) \sin \frac{n\pi x}{l} - c^2 \frac{\partial^2}{\partial x^2} \sum_{n=1}^{\infty} a_n(t) \sin \frac{n\pi x}{l} &= Q(x, t)\end{aligned}$$

Because v satisfies homogeneous boundary conditions and v , $\partial v/\partial x$, and $\partial v/\partial t$ are continuous (reasonable assumptions for the displacement of a homogeneous elastic string), the two series can in fact be differentiated term by term.

$$\sum_{n=1}^{\infty} \frac{d^2 a_n}{dt^2} \sin \frac{n\pi x}{l} - c^2 \sum_{n=1}^{\infty} a_n(t) \frac{d^2}{dx^2} \sin \frac{n\pi x}{l} = Q(x, t)$$

The operator applied to the eigenfunction is equal to the eigenvalue times the eigenfunction.

$$\sum_{n=1}^{\infty} \frac{d^2 a_n}{dt^2} \sin \frac{n\pi x}{l} - c^2 \sum_{n=1}^{\infty} a_n(t) \lambda_n \sin \frac{n\pi x}{l} = Q(x, t)$$

Combine the series on the left side.

$$\sum_{n=1}^{\infty} \left[\frac{d^2 a_n}{dt^2} - c^2 \lambda_n a_n(t) \right] \sin \frac{n\pi x}{l} = Q(x, t)$$

The left side is essentially a Fourier sine series expansion of $Q(x, t)$. To solve for the term in square brackets, multiply both sides by $\sin(m\pi x/l)$, where m is an integer,

$$\sum_{n=1}^{\infty} \left[\frac{d^2 a_n}{dt^2} - c^2 \lambda_n a_n(t) \right] \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} = Q(x, t) \sin \frac{m\pi x}{l}$$

and then integrate both sides with respect to x from 0 to l .

$$\int_0^l \sum_{n=1}^{\infty} \left[\frac{d^2 a_n}{dt^2} - c^2 \lambda_n a_n(t) \right] \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx = \int_0^l Q(x, t) \sin \frac{m\pi x}{l} dx$$

Bring the functions of t in front of the integral on the left side and substitute $Q(x, t)$ on the right side.

$$\sum_{n=1}^{\infty} \left[\frac{d^2 a_n}{dt^2} - c^2 \lambda_n a_n(t) \right] \int_0^l \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx = \int_0^l \left[F(x) - \frac{l-x}{l} H\omega^2 - \frac{x}{l} K\omega^2 \right] \cos \omega t \sin \frac{m\pi x}{l} dx$$

Since the eigenfunctions are orthogonal, the integral on the left side is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for one: $n = m$. Split up the integral on the right side and factor $\cos \omega t$.

$$\begin{aligned} & \left[\frac{d^2 a_n}{dt^2} - c^2 \lambda_n a_n(t) \right] \int_0^l \sin^2 \frac{n\pi x}{l} dx \\ &= \cos \omega t \left[\int_0^l F(x) \sin \frac{n\pi x}{l} dx - \frac{H\omega^2}{l} \int_0^l (l-x) \sin \frac{n\pi x}{l} dx - \frac{K\omega^2}{l} \int_0^l x \sin \frac{n\pi x}{l} dx \right] \end{aligned}$$

Evaluate the integrals that can be evaluated.

$$\left[\frac{d^2 a_n}{dt^2} - c^2 \lambda_n a_n(t) \right] \cdot \frac{l}{2} = \cos \omega t \left\{ \int_0^l F(x) \sin \frac{n\pi x}{l} dx - \frac{H\omega^2}{l} \left(\frac{l^2}{n\pi} \right) - \frac{K\omega^2}{l} \left[-\frac{(-1)^n l^2}{n\pi} \right] \right\}$$

Multiply both sides by $2/l$ and replace λ_n with $-(n\pi/l)^2$.

$$\begin{aligned} \frac{d^2 a_n}{dt^2} + c^2 \frac{n^2 \pi^2}{l^2} a_n &= \cos \omega t \left[\frac{2}{l} \int_0^l F(x) \sin \frac{n\pi x}{l} dx - \frac{2H\omega^2}{n\pi} + \frac{2K\omega^2(-1)^n}{n\pi} \right] \\ &= \cos \omega t \left\{ \frac{2}{l} \int_0^l F(x) \sin \frac{n\pi x}{l} dx + \frac{2\omega^2}{n\pi} [(-1)^n K - H] \right\} \end{aligned}$$

Let the constant term in curly braces be denoted by R .

$$\frac{d^2 a_n}{dt^2} + c^2 \frac{n^2 \pi^2}{l^2} a_n = R \cos \omega t$$

With the help of the method of eigenfunction expansion, the PDE has been reduced to a second-order inhomogeneous ODE. Because the ODE is linear, the general solution is the sum of a complementary solution and a particular solution.

$$a_n = a_c + a_p$$

The complementary solution satisfies the associated homogeneous equation.

$$\frac{d^2 a_c}{dt^2} + c^2 \frac{n^2 \pi^2}{l^2} a_c = 0$$

Its general solution can be written in terms of sine and cosine.

$$a_c(t) = C_7 \cos \frac{cn\pi t}{l} + C_8 \sin \frac{cn\pi t}{l}$$

Because the inhomogeneous term is cosine and there are no odd time derivatives, the particular solution is of the form $a_p(t) = C_0 \cos \omega t$. Substitute it into the ODE that a_p satisfies to find C_0 .

$$\begin{aligned} \frac{d^2 a_p}{dt^2} + c^2 \frac{n^2 \pi^2}{l^2} a_p &= R \cos \omega t \\ (-\omega^2)C_0 \cos \omega t + c^2 \frac{n^2 \pi^2}{l^2} C_0 \cos \omega t &= R \cos \omega t \\ (-\omega^2)C_0 + c^2 \frac{n^2 \pi^2}{l^2} C_0 &= R \\ C_0 \frac{c^2 n^2 \pi^2 - \omega^2 l^2}{l^2} &= R \\ C_0 &= \frac{Rl^2}{c^2 n^2 \pi^2 - \omega^2 l^2} \end{aligned}$$

Consequently, the particular solution is

$$a_p(t) = \frac{Rl^2}{c^2 n^2 \pi^2 - \omega^2 l^2} \cos \omega t.$$

Hence, the general solution for a_n is

$$a_n(t) = C_7 \cos \frac{cn\pi t}{l} + C_8 \sin \frac{cn\pi t}{l} + \frac{Rl^2}{c^2 n^2 \pi^2 - \omega^2 l^2} \cos \omega t.$$

Use the initial conditions for v in combination with the eigenfunction expansion to determine those for a_n .

$$v(x, 0) = \sum_{n=1}^{\infty} a_n(0) \sin \frac{n\pi x}{l} = \phi(x) - \frac{l-x}{l}H - \frac{x}{l}K \quad (3)$$

$$v_t(x, 0) = \sum_{n=1}^{\infty} \frac{da_n}{dt}(0) \sin \frac{n\pi x}{l} = \psi(x) \quad (4)$$

To solve for $a_n(0)$, multiply both sides of equation (3) by $\sin(m\pi x/l)$

$$\sum_{n=1}^{\infty} a_n(0) \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} = \left[\phi(x) - \frac{l-x}{l}H - \frac{x}{l}K \right] \sin \frac{m\pi x}{l}$$

and then integrate both sides with respect to x from 0 to l .

$$\int_0^l \sum_{n=1}^{\infty} a_n(0) \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx = \int_0^l \left[\phi(x) - \frac{l-x}{l}H - \frac{x}{l}K \right] \sin \frac{m\pi x}{l} dx$$

Bring the constants in front of the integral on the left side and split up the integral on the right side.

$$\begin{aligned} \sum_{n=1}^{\infty} a_n(0) \int_0^l \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx \\ = \int_0^l \phi(x) \sin \frac{m\pi x}{l} dx - \frac{H}{l} \int_0^l (l-x) \sin \frac{m\pi x}{l} dx - \frac{K}{l} \int_0^l x \sin \frac{m\pi x}{l} dx \end{aligned}$$

Since the eigenfunctions are orthogonal, the integral on the left side is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for one: $n = m$.

$$a_n(0) \int_0^l \sin^2 \frac{n\pi x}{l} dx = \int_0^l \phi(x) \sin \frac{n\pi x}{l} dx - \frac{H}{l} \int_0^l (l-x) \sin \frac{n\pi x}{l} dx - \frac{K}{l} \int_0^l x \sin \frac{n\pi x}{l} dx$$

Evaluate the integrals that can be evaluated.

$$a_n(0) \cdot \frac{l}{2} = \int_0^l \phi(x) \sin \frac{n\pi x}{l} dx - \frac{H}{l} \left(\frac{l^2}{n\pi} \right) - \frac{K}{l} \left[-\frac{(-1)^n l^2}{n\pi} \right]$$

The first initial condition for a_n is then

$$a_n(0) = \frac{2}{l} \int_0^l \phi(x) \sin \frac{n\pi x}{l} dx + \frac{2}{n\pi} [(-1)^n K - H].$$

To obtain the second one, multiply both sides of equation (4) by $\sin(m\pi x/l)$

$$\sum_{n=1}^{\infty} \frac{da_n}{dt}(0) \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} = \psi(x) \sin \frac{m\pi x}{l}$$

and then integrate both sides with respect to x from 0 to l .

$$\int_0^l \sum_{n=1}^{\infty} \frac{da_n}{dt}(0) \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx = \int_0^l \psi(x) \sin \frac{m\pi x}{l} dx$$

Bring the constants in front of the integral on the left side.

$$\sum_{n=1}^{\infty} \frac{da_n}{dt}(0) \int_0^l \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx = \int_0^l \psi(x) \sin \frac{m\pi x}{l} dx$$

Since the eigenfunctions are orthogonal, the integral on the left side is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for one: $n = m$.

$$\frac{da_n}{dt}(0) \int_0^l \sin^2 \frac{n\pi x}{l} dx = \int_0^l \psi(x) \sin \frac{n\pi x}{l} dx$$

Evaluate the integral on the left side.

$$\frac{da_n}{dt}(0) \cdot \frac{l}{2} = \int_0^l \psi(x) \sin \frac{n\pi x}{l} dx$$

The second initial condition for a_n is then

$$\frac{da_n}{dt}(0) = \frac{2}{l} \int_0^l \psi(x) \sin \frac{n\pi x}{l} dx.$$

Now apply the two initial conditions to obtain a system of equations for C_7 and C_8 .

$$a(0) = C_7 + \frac{Rl^2}{c^2n^2\pi^2 - \omega^2l^2} = \frac{2}{l} \int_0^l \phi(x) \sin \frac{n\pi x}{l} dx + \frac{2}{n\pi} [(-1)^n K - H]$$

$$\frac{da_n}{dt}(0) = \frac{cn\pi}{l} (C_8) = \frac{2}{l} \int_0^l \psi(x) \sin \frac{n\pi x}{l} dx$$

Solving the system yields

$$C_7 = \frac{2}{l} \int_0^l \phi(x) \sin \frac{n\pi x}{l} dx + \frac{2}{n\pi} [(-1)^n K - H] - \frac{Rl^2}{c^2n^2\pi^2 - \omega^2l^2}$$

$$C_8 = \frac{2}{cn\pi} \int_0^l \psi(x) \sin \frac{n\pi x}{l} dx.$$

Plugging in the formulas for C_7 and C_8 , the general solution for $a_n(t)$ becomes

$$a_n(t) = \left\{ \frac{2}{l} \int_0^l \phi(s) \sin \frac{n\pi s}{l} ds + \frac{2}{n\pi} [(-1)^n K - H] - \frac{Rl^2}{c^2n^2\pi^2 - \omega^2l^2} \right\} \cos \frac{cn\pi t}{l}$$

$$+ \left[\frac{2}{cn\pi} \int_0^l \psi(s) \sin \frac{n\pi s}{l} ds \right] \sin \frac{cn\pi t}{l}$$

$$+ \frac{Rl^2}{c^2n^2\pi^2 - \omega^2l^2} \cos \omega t,$$

where

$$R = \frac{2}{l} \int_0^l F(s) \sin \frac{n\pi s}{l} ds + \frac{2\omega^2}{n\pi} [(-1)^n K - H].$$

Therefore, since $u(x, t) = r(x, t) + v(x, t)$,

$$u(x, t) = \frac{l-x}{l} H \cos \omega t + \frac{x}{l} K \cos \omega t + \sum_{n=1}^{\infty} a_n(t) \sin \frac{n\pi x}{l}, \quad 0 \leq x \leq l.$$

Method 2 - Without Using Term-by-Term Differentiation

The eigenfunctions of the Helmholtz equation are known to form a complete set, so all of the functions in the PDE for v can be expanded in terms of them.

$$v(x, t) = \sum_{n=1}^{\infty} A_n(t) \zeta_n(x) \quad \rightarrow \quad v \zeta_m = \sum_{n=1}^{\infty} A_n \zeta_n \zeta_m \quad \rightarrow \quad \int_0^l v \zeta_n dx = A_n \int_0^l \zeta_n^2 dx = A_n \cdot \frac{l}{2}$$

$$\frac{\partial^2 v}{\partial t^2} = \sum_{n=1}^{\infty} B_n(t) \zeta_n(x) \quad \rightarrow \quad \frac{\partial^2 v}{\partial t^2} \zeta_m = \sum_{n=1}^{\infty} B_n \zeta_n \zeta_m \quad \rightarrow \quad \int_0^l \frac{\partial^2 v}{\partial t^2} \zeta_n dx = B_n \int_0^l \zeta_n^2 dx = B_n \cdot \frac{l}{2}$$

$$Q(x, t) = \sum_{n=1}^{\infty} D_n(t) \zeta_n(x) \quad \rightarrow \quad Q(x, t) \zeta_m = \sum_{n=1}^{\infty} D_n \zeta_n \zeta_m \quad \rightarrow \quad \int_0^l Q(x, t) \zeta_n dx = D_n \int_0^l \zeta_n^2 dx = D_n \cdot \frac{l}{2}$$

$$\frac{\partial^2 v}{\partial x^2} = \sum_{n=1}^{\infty} E_n(t) \zeta_n(x) \quad \rightarrow \quad \frac{\partial^2 v}{\partial x^2} \zeta_m = \sum_{n=1}^{\infty} E_n \zeta_n \zeta_m \quad \rightarrow \quad \int_0^l \frac{\partial^2 v}{\partial x^2} \zeta_n dx = E_n \int_0^l \zeta_n^2 dx = E_n \cdot \frac{l}{2}$$

It should be emphasized that these are generalized Fourier series expansions for the functions, not product solutions that come about from using the method of separation of variables. Solve the latter equations for the generalized Fourier coefficients.

$$\begin{aligned}
 A_n(t) &= \frac{2}{l} \int_0^l v \zeta_n dx \\
 B_n(t) &= \frac{2}{l} \int_0^l \frac{\partial^2 v}{\partial t^2} \zeta_n dx = \frac{d^2}{dt^2} \left(\frac{2}{l} \int_0^l v \zeta_n dx \right) = \frac{d^2 A_n}{dt^2} \\
 D_n(t) &= \frac{2}{l} \int_0^l Q(x, t) \zeta_n dx \\
 E_n(t) &= \frac{2}{l} \int_0^l \frac{\partial^2 v}{\partial x^2} \zeta_n dx = \frac{2}{l} \underbrace{\left(\frac{\partial v}{\partial x} \zeta_n \Big|_0^l - \int_0^l \frac{\partial v}{\partial x} \frac{d\zeta_n}{dx} dx \right)}_{=0} = -\frac{2n\pi}{l^2} \int_0^l \frac{\partial v}{\partial x} \cos \frac{n\pi x}{l} dx
 \end{aligned}$$

Apply integration by parts once more in order to write E_n in terms of A_n .

$$\begin{aligned}
 E_n(t) &= -\frac{2n\pi}{l^2} \left[\underbrace{v \cos \frac{n\pi x}{l}}_{=0} \Big|_0^l - \int_0^l v \left(-\frac{n\pi}{l} \sin \frac{n\pi x}{l} \right) dx \right] \\
 &= -\frac{n^2 \pi^2}{l^2} \left(\frac{2}{l} \int_0^l v \sin \frac{n\pi x}{l} dx \right) \\
 &= -\frac{n^2 \pi^2}{l^2} A_n
 \end{aligned}$$

Now that the coefficients are known, substitute the eigenfunction expansions into the PDE.

$$\begin{aligned}
 v_{tt} - c^2 v_{xx} &= Q(x, t) \\
 \sum_{n=1}^{\infty} B_n(t) \zeta_n(x) - c^2 \sum_{n=1}^{\infty} E_n(t) \zeta_n(x) &= \sum_{n=1}^{\infty} D_n(t) \zeta_n(x) \\
 \sum_{n=1}^{\infty} [B_n(t) - c^2 E_n(t)] \zeta_n(x) &= \sum_{n=1}^{\infty} D_n(t) \zeta_n(x)
 \end{aligned}$$

Thus,

$$B_n(t) - c^2 E_n(t) = D_n(t).$$

Substitute the formulas for B_n , E_n , and D_n to obtain an ODE for A_n exclusively.

$$\begin{aligned}
 \frac{d^2 A_n}{dt^2} + c^2 \frac{n^2 \pi^2}{l^2} A_n &= \frac{2}{l} \int_0^l Q(x, t) \zeta_n dx \\
 &= \frac{2}{l} \int_0^l \left[F(x) - \frac{l-x}{l} H\omega^2 - \frac{x}{l} K\omega^2 \right] \cos \omega t \sin \frac{n\pi x}{l} dx \\
 &= \left\{ \frac{2}{l} \int_0^l \left[F(x) - \frac{l-x}{l} H\omega^2 - \frac{x}{l} K\omega^2 \right] \sin \frac{n\pi x}{l} dx \right\} \cos \omega t \\
 &= R \cos \omega t
 \end{aligned}$$

This is the same ODE that was obtained for a_n in Method 1. The initial conditions are also the same as before, so $A_n(t) = a_n(t)$ and the same solution is obtained for v and u .

Method 3 - Mr. Strauss's Way

Here the method of eigenfunction expansion will be applied directly to the inhomogeneous PDE for u . The eigenfunctions of the Helmholtz equation are known to form a complete set, so all the functions in the PDE can be expanded in terms of them.

$$\begin{aligned}
 u(x, t) &= \sum_{n=1}^{\infty} b_n(t) \zeta_n(x) &\rightarrow u \zeta_m &= \sum_{n=1}^{\infty} b_n \zeta_n \zeta_m &\rightarrow \int_0^l u \zeta_n dx = b_n \int_0^l \zeta_n^2 dx = b_n \cdot \frac{l}{2} \\
 \frac{\partial u}{\partial t} &= \sum_{n=1}^{\infty} J_n(t) \zeta_n(x) &\rightarrow \frac{\partial u}{\partial t} \zeta_m &= \sum_{n=1}^{\infty} J_n \zeta_n \zeta_m &\rightarrow \int_0^l \frac{\partial u}{\partial t} \zeta_n dx = J_n \int_0^l \zeta_n^2 dx = J_n \cdot \frac{l}{2} \\
 \frac{\partial^2 u}{\partial t^2} &= \sum_{n=1}^{\infty} e_n(t) \zeta_n(x) &\rightarrow \frac{\partial^2 u}{\partial t^2} \zeta_m &= \sum_{n=1}^{\infty} e_n \zeta_n \zeta_m &\rightarrow \int_0^l \frac{\partial^2 u}{\partial t^2} \zeta_n dx = e_n \int_0^l \zeta_n^2 dx = e_n \cdot \frac{l}{2} \\
 F(x) \cos \omega t &= \sum_{n=1}^{\infty} L_n(t) \zeta_n(x) &\rightarrow F(x) \cos \omega t \zeta_m &= \sum_{n=1}^{\infty} L_n \zeta_n \zeta_m &\rightarrow \cos \omega t \int_0^l F(x) \zeta_n dx = L_n \int_0^l \zeta_n^2 dx = L_n \cdot \frac{l}{2} \\
 \frac{\partial^2 u}{\partial x^2} &= \sum_{n=1}^{\infty} G_n(t) \zeta_n(x) &\rightarrow \frac{\partial^2 u}{\partial x^2} \zeta_m &= \sum_{n=1}^{\infty} G_n \zeta_n \zeta_m &\rightarrow \int_0^l \frac{\partial^2 u}{\partial x^2} \zeta_n dx = G_n \int_0^l \zeta_n^2 dx = G_n \cdot \frac{l}{2}
 \end{aligned}$$

It should be emphasized that these are generalized Fourier series expansions for the functions, not product solutions that come about from using the method of separation of variables. Solve the latter equations for the generalized Fourier coefficients.

$$\begin{aligned}
 b_n(t) &= \frac{2}{l} \int_0^l u \zeta_n dx \\
 J_n(t) &= \frac{2}{l} \int_0^l \frac{\partial u}{\partial t} \zeta_n dx = \frac{d}{dt} \left(\frac{2}{l} \int_0^l u \zeta_n dx \right) = \frac{db_n}{dt} \\
 e_n(t) &= \frac{2}{l} \int_0^l \frac{\partial^2 u}{\partial t^2} \zeta_n dx = \frac{d^2}{dt^2} \left(\frac{2}{l} \int_0^l u \zeta_n dx \right) = \frac{d^2 b_n}{dt^2} \\
 L_n(t) &= \frac{2}{l} \cos \omega t \int_0^l F(x) \zeta_n dx \\
 G_n(t) &= \frac{2}{l} \int_0^l \frac{\partial^2 u}{\partial x^2} \zeta_n dx = \frac{2}{l} \left(\underbrace{\frac{\partial u}{\partial x} \zeta_n}_0 \Big|_0^l - \int_0^l \frac{\partial u}{\partial x} \frac{d\zeta_n}{dx} dx \right) = -\frac{2n\pi}{l^2} \int_0^l \frac{\partial u}{\partial x} \cos \frac{n\pi x}{l} dx
 \end{aligned}$$

Apply integration by parts once more in order to write G_n in terms of b_n .

$$\begin{aligned}
 G_n(t) &= -\frac{2n\pi}{l^2} \left[u \cos \frac{n\pi x}{l} \Big|_0^l - \int_0^l u \left(-\frac{n\pi}{l} \sin \frac{n\pi x}{l} \right) dx \right] \\
 &= -\frac{2n\pi}{l^2} \left[u(l, t) \cos n\pi - u(0, t) + \frac{n\pi}{l} \int_0^l u \sin \frac{n\pi x}{l} dx \right] \\
 &= -\frac{2n\pi}{l^2} \left[(-1)^n K \cos \omega t - H \cos \omega t + \frac{n\pi}{2} \cdot \frac{2}{l} \int_0^l u \sin \frac{n\pi x}{l} dx \right] \\
 &= -\frac{2n\pi}{l^2} \left[(-1)^n K \cos \omega t - H \cos \omega t + \frac{n\pi}{2} b_n \right]
 \end{aligned}$$

Now that the coefficients are known, substitute the eigenfunction expansions into the PDE.

$$\begin{aligned}
 u_{tt} - c^2 u_{xx} &= F(x) \cos \omega t \\
 \sum_{n=1}^{\infty} e_n(t) \zeta_n(x) - c^2 \sum_{n=1}^{\infty} G_n(t) \zeta_n(x) &= \sum_{n=1}^{\infty} L_n(t) \zeta_n(x) \\
 \sum_{n=1}^{\infty} [e_n(t) - c^2 G_n(t)] \zeta_n(x) &= \sum_{n=1}^{\infty} L_n(t) \zeta_n(x)
 \end{aligned}$$

Thus,

$$e_n(t) - c^2 G_n(t) = L_n(t).$$

Substitute the formulas for e_n , G_n , and L_n to obtain an ODE for b_n exclusively.

$$\frac{d^2 b_n}{dt^2} + c^2 \frac{2n\pi}{l^2} \left[(-1)^n K \cos \omega t - H \cos \omega t + \frac{n\pi}{2} b_n \right] = \frac{2}{l} \cos \omega t \int_0^l F(x) \zeta_n dx$$

Replace ζ_n with $\sin(n\pi x/l)$ and bring the terms without b_n to the right side.

$$\begin{aligned}
 \frac{d^2 b_n}{dt^2} + c^2 \frac{n^2 \pi^2}{l^2} b_n &= \frac{2}{l} \cos \omega t \int_0^l F(x) \sin \frac{n\pi x}{l} dx - c^2 \frac{2n\pi}{l^2} [(-1)^n K \cos \omega t - H \cos \omega t] \\
 &= \left\{ \frac{2}{l} \int_0^l F(x) \sin \frac{n\pi x}{l} dx - 2n\pi \frac{c^2}{l^2} [(-1)^n K - H] \right\} \cos \omega t
 \end{aligned}$$

Let the term in curly braces be denoted as S .

$$\frac{d^2 b_n}{dt^2} + c^2 \frac{n^2 \pi^2}{l^2} b_n = S \cos \omega t$$

This is essentially the same ODE that a_n satisfies—only the coefficient of cosine is different, so the same general solution can be used here.

$$b_n(t) = C_9 \cos \frac{cn\pi t}{l} + C_{10} \sin \frac{cn\pi t}{l} + \frac{Sl^2}{c^2 n^2 \pi^2 - \omega^2 l^2} \cos \omega t$$

Use the initial conditions for u in combination with the eigenfunction expansion to determine those for b_n .

$$\begin{aligned}
 u(x, 0) = \sum_{n=1}^{\infty} b_n(0) \zeta_n(x) = \phi(x) &\rightarrow \phi(x) \zeta_m = \sum_{n=1}^{\infty} b_n(0) \zeta_n \zeta_m &\rightarrow \int_0^l \phi(x) \zeta_n dx = b_n(0) \int_0^l \zeta_n^2 dx = b_n(0) \cdot \frac{l}{2} \\
 u_t(x, 0) = \sum_{n=1}^{\infty} J_n(0) \zeta_n(x) = \psi(x) &\rightarrow \psi(x) \zeta_m = \sum_{n=1}^{\infty} \frac{db_n}{dt}(0) \zeta_n \zeta_m &\rightarrow \int_0^l \psi(x) \zeta_n dx = \frac{db_n}{dt}(0) \int_0^l \zeta_n^2 dx = \frac{db_n}{dt}(0) \cdot \frac{l}{2}
 \end{aligned}$$

Solve the latter equations for the initial conditions.

$$\begin{aligned}
 b_n(0) &= \frac{2}{l} \int_0^l \phi(x) \zeta_n dx \\
 \frac{db_n}{dt}(0) &= \frac{2}{l} \int_0^l \psi(x) \zeta_n dx
 \end{aligned}$$

Now apply them to obtain a system of equations for C_9 and C_{10} .

$$b_n(0) = C_9 + \frac{Sl^2}{c^2n^2\pi^2 - \omega^2l^2} = \frac{2}{l} \int_0^l \phi(x) \sin \frac{n\pi x}{l} dx$$

$$\frac{db_n}{dt}(0) = \frac{cn\pi}{l}(C_{10}) = \frac{2}{l} \int_0^l \psi(x) \sin \frac{n\pi x}{l} dx$$

Solving this system yields

$$C_9 = \frac{2}{l} \int_0^l \phi(x) \sin \frac{n\pi x}{l} dx - \frac{Sl^2}{c^2n^2\pi^2 - \omega^2l^2}$$

$$C_{10} = \frac{2}{cn\pi} \int_0^l \psi(x) \sin \frac{n\pi x}{l} dx.$$

Plugging in the formulas for C_9 and C_{10} , the general solution for $b_n(t)$ becomes

$$b_n(t) = \left\{ \frac{2}{l} \int_0^l \phi(s) \sin \frac{n\pi s}{l} ds - \frac{Sl^2}{c^2n^2\pi^2 - \omega^2l^2} \right\} \cos \frac{cn\pi t}{l}$$

$$+ \left[\frac{2}{cn\pi} \int_0^l \psi(s) \sin \frac{n\pi s}{l} ds \right] \sin \frac{cn\pi t}{l}$$

$$+ \frac{Sl^2}{c^2n^2\pi^2 - \omega^2l^2} \cos \omega t,$$

where

$$S = \frac{2}{l} \int_0^l F(s) \sin \frac{n\pi s}{l} ds - 2n\pi \frac{c^2}{l^2} [(-1)^n K - H].$$

Therefore,

$$u(x, t) = \sum_{n=1}^{\infty} b_n(t) \sin \frac{n\pi x}{l}, \quad 0 < x < l.$$

This solution does not satisfy the PDE at $x = 0$ and $x = l$ because $u(0, t) \neq \zeta(0) = 0$ and $u(l, t) \neq \zeta(l) = 0$. Also, it converges more slowly than the solution obtained using Method 1 and Method 2.