Exercise 13

If friction is present, the wave equation takes the form

$$u_{tt} - c^2 u_{xx} = -ru_t,$$

where the resistance $r > 0$ is a constant. Consider a periodic source at one end: $u(0, t) = 0$, $u(l, t) = Ae^{i\omega t}$.

(a) Show that the PDE and the BC are satisfied by

$$\mathscr{U}(x, t) = Ae^{i\omega t}\frac{\sin\beta x}{\sin\beta l}, \quad \text{where } \beta^2 c^2 = \omega^2 - ir\omega.$$

(b) No matter what the IC, $u(x, 0)$ and $u_t(x, 0)$, are, show that $\mathscr{U}(x, t)$ is the asymptotic form of the solution $u(x, t)$ as $t \to \infty$.

(c) Show that you can get resonance as $r \to 0$ if $\omega = m\pi c/l$ for some integer m.

(d) Show that friction can prevent resonance from occuring.

Solution

Part (a)

Substitute the given function for $\mathscr{U}$ into the PDE to see if it is a solution.

$$\frac{\partial^2}{\partial t^2}\left(Ae^{i\omega t}\frac{\sin\beta x}{\sin\beta l}\right) - c^2\frac{\partial^2}{\partial x^2}\left(Ae^{i\omega t}\frac{\sin\beta x}{\sin\beta l}\right) = -r\frac{\partial}{\partial t}\left(Ae^{i\omega t}\frac{\sin\beta x}{\sin\beta l}\right)$$

Evaluate the derivatives.

$$A(i\omega)^2 e^{i\omega t}\frac{\sin\beta x}{\sin\beta l} - c^2 A(-\beta^2)e^{i\omega t}\frac{\sin\beta x}{\sin\beta l} = -rA(i\omega)e^{i\omega t}\frac{\sin\beta x}{\sin\beta l}$$

Divide both sides by $\mathscr{U}$.

$$(i\omega)^2 - c^2(-\beta^2) = -r(i\omega)$$
$$-\omega^2 + \beta^2 c^2 = -ir\omega$$

Substitute $\beta^2 c^2 = \omega^2 - ir\omega$ here.

$$-\cancel{\omega^2} + \cancel{\omega^2} - ir\omega = -ir\omega$$

Therefore, $\mathscr{U}$ satisfies the PDE.

$$\mathscr{U}(0, t) = Ae^{i\omega t}\frac{\sin\beta 0}{\sin\beta l} = 0$$
$$\mathscr{U}(l, t) = Ae^{i\omega t}\frac{\cancel{\sin\beta l}}{\cancel{\sin\beta l}} = Ae^{i\omega t}$$

It also satisfies the boundary conditions.

Part (b)

In order to make the boundary condition at $x = l$ homogeneous, we make the substitution $u(x,t) = v(x,t) + \mathscr{U}(x,t)$. As a result, the PDE becomes

$$u_{tt} - c^2 u_{xx} = -ru_t \quad \to \quad \begin{aligned} [v_{tt} + \mathscr{U}_{tt}] - c^2[v_{xx} + \mathscr{U}_{xx}] &= -r[v_t + \mathscr{U}_t] \\ v_{tt} - c^2 v_{xx} + \mathscr{U}_{tt} - c^2 \mathscr{U}_{xx} &= -rv_t - r\mathscr{U}_t \end{aligned}$$

Since $\mathscr{U}$ was shown to satisfy the PDE, $\mathscr{U}_{tt} - c^2\mathscr{U}_{xx} = -r\mathscr{U}_t$, and the equation reduces to

$$v_{tt} - c^2 v_{xx} = -rv_t.$$

The boundary conditions associated with it are homogeneous as desired.

$$\begin{aligned} v(0,t) &= u(0,t) - \mathscr{U}(0,t) = 0 - 0 = 0 \\ v(l,t) &= u(l,t) - \mathscr{U}(l,t) = Ae^{i\omega t} - Ae^{i\omega t} = 0 \end{aligned}$$

Because the PDE and its boundary conditions are linear and homogeneous, the method of separation of variables can be applied to solve the equation. Assume a product solution of the form $v(x,t) = X(x)T(t)$ and plug it into the PDE

$$v_{tt} - c^2 v_{xx} = -rv_t \quad \to \quad XT'' - c^2 X''T = -rXT'$$

and the boundary conditions.

$$\begin{aligned} v(0,t) = 0 \quad &\to \quad X(0)T(t) = 0 \quad \to \quad X(0) = 0 \\ v(l,t) = 0 \quad &\to \quad X(l)T(t) = 0 \quad \to \quad X(l) = 0 \end{aligned}$$

Now separate variables in the PDE: bring the constants and functions of t to the left side and the functions of x to the right side.

$$XT'' + rXT' = c^2 X''T \quad \to \quad \frac{T'' + rT'}{c^2 T} = \frac{X''}{X}$$

The only way a function of t can be equal to a function of x is if both are equal to a constant λ.

$$\frac{T'' + rT'}{c^2 T} = \frac{X''}{X} = \lambda$$

Values of λ for which the boundary conditions are satisfied are called the eigenvalues, and the nontrivial functions associated with them are known as the eigenfunctions.

Determination of Positive Eigenvalues: $\lambda = \mu^2$

Suppose that λ is positive. Then the ODE for X becomes

$$\frac{X''}{X} = \mu^2.$$

Multiply both sides by X.

$$X'' = \mu^2 X$$

The solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \mu x + C_2 \sinh \mu x$$

Apply the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} X(0) &= C_1 = 0 \\ X(l) &= C_1 \cosh \mu l + C_2 \sinh \mu l = 0 \end{aligned}$$

Since $C_1 = 0$, the second equation reduces to $C_2 \sinh \mu l = 0$. Hyperbolic sine is not oscillatory, so the only way this equation is satisfied is if $C_2 = 0$. The trivial solution is obtained, so there are no positive eigenvalues.

Determination of the Zero Eigenvalue: $\lambda = 0$

Suppose that λ is zero. Then the ODE for X becomes

$$\frac{X''}{X} = 0.$$

Multiply both sides by X.

$$X'' = 0$$

The general solution is obtained by integrating both sides with respect to x twice.

$$X(x) = C_3 x + C_4$$

Apply the boundary conditions to determine C_3 and C_4.

$$\begin{aligned} X(0) &= C_4 = 0 \\ X(l) &= C_3 l + C_4 = 0 \end{aligned}$$

Since $C_4 = 0$, the second equation reduces to $C_3 = 0$. The trivial solution is obtained, so zero is not an eigenvalue.

Determination of Negative Eigenvalues: $\lambda = -\gamma^2$

Suppose that λ is negative. Then the ODE for X becomes

$$\frac{X''}{X} = -\gamma^2.$$

Multiply both sides by X.

$$X'' = -\gamma^2 X$$

Its solution can be written in terms of sine and cosine.

$$X(x) = C_5 \cos \gamma x + C_6 \sin \gamma x$$

Apply the boundary conditions to determine C_5 and C_6.

$$\begin{aligned} X(0) &= C_5 = 0 \\ X(l) &= C_5 \cos \gamma l + C_6 \sin \gamma l = 0 \end{aligned}$$

Since $C_5 = 0$, the second equation reduces to $C_6 \sin \gamma l = 0$. To avoid getting the trivial solution, we insist that $C_6 \neq 0$. Then

$$\begin{aligned} \sin \gamma l &= 0 \\ \gamma l &= n\pi, \quad n = 1, 2, \ldots \\ \gamma_n &= \frac{n\pi}{l}, \quad n = 1, 2, \ldots. \end{aligned}$$

The eigenfunctions associated with these eigenvalues for λ are

$$X(x) = C_6 \sin \gamma x \quad \rightarrow \quad X_n(x) = \sin \frac{n\pi x}{l}, \quad n = 1, 2, \ldots.$$

Now the related ODE for T will be solved with $\lambda = -(n\pi/l)^2$.

$$\frac{T'' + rT'}{c^2 T} = -\frac{n^2\pi^2}{l^2}$$

Multiply both sides by $c^2 T$ and bring all terms to the left side.

$$T'' + rT' + \frac{c^2 n^2 \pi^2}{l^2} T = 0$$

Since the coefficients are constant in time, the solution is of the form $T = e^{pt}$. Take two derivatives of it

$$T = e^{pt} \quad \rightarrow \quad T' = pe^{pt} \quad \rightarrow \quad T'' = p^2 e^{pt}$$

and plug it into the ODE to find p.

$$p^2 e^{pt} + rpe^{pt} + \frac{c^2 n^2 \pi^2}{l^2} e^{pt} = 0$$

$$p^2 + rp + \frac{c^2 n^2 \pi^2}{l^2} = 0$$

$$p = \frac{-r \pm \sqrt{r^2 - 4\frac{c^2 n^2 \pi^2}{l^2}}}{2} = -\frac{r}{2} \pm \frac{1}{2}\sqrt{r^2 - \left(\frac{2\pi c}{l} n\right)^2}$$

Depending on how large the friction parameter r is, p can either be real or complex.

$$\begin{aligned} &\text{Case I:} && r = \frac{2\pi c}{l} n && \text{or} && n = \frac{rl}{2\pi c} && \Rightarrow && p = -\frac{r}{2} \quad \text{(repeated)} \\ &\text{Case II:} && r > \frac{2\pi c}{l} n && \text{or} && n < \frac{rl}{2\pi c} && \Rightarrow && p = -\frac{r}{2} \pm \frac{1}{2}\sqrt{r^2 - \left(\frac{2\pi c}{l} n\right)^2} \\ &\text{Case III:} && r < \frac{2\pi c}{l} n && \text{or} && n > \frac{rl}{2\pi c} && \Rightarrow && p = -\frac{r}{2} \pm \frac{i}{2}\sqrt{\left(\frac{2\pi c}{l} n\right)^2 - r^2} \end{aligned}$$

The solution thus has three possible forms.

$$\begin{aligned} &\text{Case I:} \quad T_{n_1}(t) = \exp\left(-\frac{r}{2}t\right)(C_7 + C_8 t) \\ &\text{Case II:} \quad T_{n_2}(t) = \exp\left(-\frac{r}{2}t\right)\left\{C_9 \exp\left[\frac{t}{2}\sqrt{r^2 - \left(\frac{2\pi c}{l} n\right)^2}\right] + C_{10} \exp\left[-\frac{t}{2}\sqrt{r^2 - \left(\frac{2\pi c}{l} n\right)^2}\right]\right\} \\ &\text{Case III:} \quad T_{n_3}(t) = \exp\left(-\frac{r}{2}t\right)\left\{C_{11} \cos\left[\frac{t}{2}\sqrt{\left(\frac{2\pi c}{l} n\right)^2 - r^2}\right] + C_{12} \sin\left[\frac{t}{2}\sqrt{\left(\frac{2\pi c}{l} n\right)^2 - r^2}\right]\right\} \end{aligned}$$

Because of $e^{-rt/2}$, $T_{n_1}(t)$, $T_{n_2}(t)$, and $T_{n_3}(t)$ tend to zero as $t \to \infty$. According to the principle of superposition, the general solution to the PDE for v is a linear combination of the eigenfunctions over all the eigenvalues.

$$v(x,t) = \sum_{n=1}^{\infty} X_n(x) T_n(t)$$

Depending on the magnitude of the ratio $rl/(2\pi c)$ and whether it is a positive integer or not, $T_{n_1}(t)$ and $T_{n_2}(t)$ may or may not be a part of the solution.

$$v(x,t) = \begin{cases} \sum_{n=1}^{\infty} T_{n_3}(t) \sin \frac{n\pi x}{l} & \text{if } \frac{rl}{2\pi c} < 1 \\ T_{n_1}(t) \sin \frac{n\pi x}{l}\Big|_{n=\frac{rl}{2\pi c}=1} + \sum_{n=2}^{\infty} T_{n_3}(t) \sin \frac{n\pi x}{l} & \text{if } \frac{rl}{2\pi c} = 1 \\ \sum_{1 \le n < \frac{rl}{2\pi c}}^{\infty} T_{n_2}(t) \sin \frac{n\pi x}{l} + \sum_{\frac{rl}{2\pi c} < n < \infty}^{\infty} T_{n_3}(t) \sin \frac{n\pi x}{l} & \text{if } \frac{rl}{2\pi c} > 1 \text{ and } \frac{rl}{2\pi c} \notin \mathbb{Z}^+ \\ \sum_{1 \le n < \frac{rl}{2\pi c}}^{\infty} T_{n_2}(t) \sin \frac{n\pi x}{l} + T_{n_1}(t) \sin \frac{n\pi x}{l}\Big|_{n=\frac{rl}{2\pi c}} + \sum_{\frac{rl}{2\pi c} < n < \infty}^{\infty} T_{n_3}(t) \sin \frac{n\pi x}{l} & \text{if } \frac{rl}{2\pi c} > 1 \text{ and } \frac{rl}{2\pi c} \in \mathbb{Z}^+ \end{cases}$$

Since $u(x,t) = v(x,t) + \mathscr{U}(x,t)$,

$$\lim_{t\to\infty} u(x,t) = \underbrace{\lim_{t\to\infty} v(x,t)}_{=0} + \lim_{t\to\infty} \mathscr{U}(x,t) = \lim_{t\to\infty} \mathscr{U}(x,t).$$

Therefore, $\mathscr{U}(x,t)$ is the asymptotic form of the solution as $t \to \infty$. Note that giving the initial conditions, $u(x,0)$ and $u_t(x,0)$, will only determine the constants C_7, C_8, C_9, C_{10}, C_{11}, and C_{12}; they will not affect the behavior of $T_n(t)$ as $t \to \infty$.

Part (c)

If $r \to 0$, then the first condition for $v(x,t)$ applies

$$\lim_{r\to 0} v(x,t) = \sum_{n=1}^{\infty} T_{n_3}(t) \sin \frac{n\pi x}{l},$$

and $T_{n_3}(t)$ simplifies to

$$\begin{aligned} \lim_{r\to 0} T_{n_3}(t) &= C_{11} \cos\left[\frac{t}{2}\left(\frac{2\pi c}{l} n\right)\right] + C_{12} \sin\left[\frac{t}{2}\left(\frac{2\pi c}{l} n\right)\right] \\ &= C_{11} \cos\left(\frac{cn\pi}{l} t\right) + C_{12} \sin\left(\frac{cn\pi}{l} t\right). \end{aligned}$$

Nothing blows up here, so now we turn our attention to $\mathscr{U}$. If $r \to 0$, then

$$\beta^2 c^2 = \omega^2 - \underbrace{ir\omega}_{\to 0} \quad \to \quad \beta = \frac{\omega}{c},$$

and

$$\begin{aligned} \lim_{r\to 0} \mathscr{U}(x,t) &= \lim_{r\to 0} A e^{i\omega t} \frac{\sin \beta x}{\sin \beta l} \\ &= A e^{i\omega t} \frac{\sin \frac{\omega x}{c}}{\sin \frac{\omega l}{c}}. \end{aligned}$$

This limit blows up when

$$\begin{aligned} \sin\frac{\omega l}{c} &= 0 \\ \frac{\omega l}{c} &= m\pi, \quad m = 1, 2, \ldots \\ \omega &= \frac{m\pi c}{l}, \quad m = 1, 2, \ldots. \end{aligned}$$

Therefore, resonance does occur for these values of ω if there is no friction.

Part (d)

Assuming that friction is present implies that $r > 0$.

$$\beta^2 c^2 = \omega^2 - ir\omega \quad \to \quad \beta^2 = \frac{\omega^2 - ir\omega}{c^2} \quad \to \quad \beta = \frac{\sqrt{\omega^2 - ir\omega}}{c}$$

For a complex number s, the square root function can be written in terms of the logarithm.

$$\sqrt{s} = \exp\left(\frac{1}{2}\log s\right)$$

The principal branch of $\sqrt{s}$ is obtained by taking the principal branch of $\log s$.

$$\begin{aligned} &= \exp\left(\frac{1}{2}\operatorname{Log} s\right), \quad (|s| > 0,\ -\pi < \operatorname{Arg} s < \pi) \\ &= \exp\left[\frac{1}{2}(\ln R + i\Theta)\right] \\ &= \sqrt{R}e^{i\Theta/2}, \end{aligned}$$

where $R = |s|$ is the magnitude of s and $\Theta = \operatorname{Arg} s$ is the principal argument of s. In other words, this expression for $\sqrt{s}$ can be used for a complex number $s = Re^{i\Theta}$. For $s = \omega^2 - ir\omega$, we have

$$R = \sqrt{(\omega^2)^2 + (-r\omega)^2} = \sqrt{\omega^4 + r^2\omega^2} \quad \text{and} \quad \Theta = \tan^{-1}\frac{-r\omega}{\omega^2} = -\tan^{-1}\frac{r}{\omega}.$$

So then

$$\sqrt{\omega^2 - ir\omega} = \sqrt[4]{\omega^4 + r^2\omega^2}\exp\left(-\frac{i}{2}\tan^{-1}\frac{r}{\omega}\right).$$

Apply Euler's formula here.

$$= \sqrt[4]{\omega^4 + r^2\omega^2}\left[\cos\left(\frac{1}{2}\tan^{-1}\frac{r}{\omega}\right) - i\sin\left(\frac{1}{2}\tan^{-1}\frac{r}{\omega}\right)\right]$$

Apply the half-angle formulas for sine and cosine here to remove the 1/2 from the arguments.

$$= \sqrt[4]{\omega^4 + r^2\omega^2}\left[\sqrt{\frac{1 + \cos\left(\tan^{-1}\frac{r}{\omega}\right)}{2}} - i\sqrt{\frac{1 - \cos\left(\tan^{-1}\frac{r}{\omega}\right)}{2}}\right]$$

$$\sqrt{\omega^2 - ir\omega} = \frac{\sqrt[4]{\omega^4 + r^2\omega^2}}{\sqrt{2}}\left[\sqrt{1 + \cos\left(\tan^{-1}\frac{r}{\omega}\right)} - i\sqrt{1 - \cos\left(\tan^{-1}\frac{r}{\omega}\right)}\right]$$

Draw the implied right triangle in order to determine the cosine.

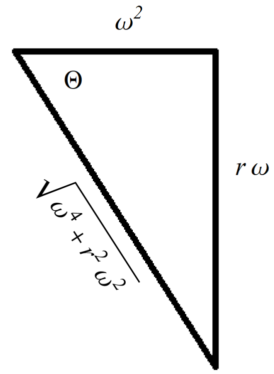

Figure 1: We see that $\cos\left(\tan^{-1}\frac{r}{\omega}\right) = \omega^2/\sqrt{\omega^4 + r^2\omega^2}$.

Consequently,

$$\begin{aligned}\sqrt{\omega^2 - ir\omega} &= \frac{\sqrt[4]{\omega^4 + r^2\omega^2}}{\sqrt{2}}\left(\sqrt{1 + \frac{\omega^2}{\sqrt{\omega^4 + r^2\omega^2}}} - i\sqrt{1 - \frac{\omega^2}{\sqrt{\omega^4 + r^2\omega^2}}}\right) \\ &= \frac{1}{\sqrt{2}}\left(\sqrt{\sqrt{\omega^4 + r^2\omega^2} + \omega^2} - i\sqrt{\sqrt{\omega^4 + r^2\omega^2} - \omega^2}\right),\end{aligned}$$

and

$$\beta = \frac{1}{c\sqrt{2}}\left(\sqrt{\sqrt{\omega^4 + r^2\omega^2} + \omega^2} - i\sqrt{\sqrt{\omega^4 + r^2\omega^2} - \omega^2}\right).$$

Resonance only occurs if $\sin\beta l$ is equal to zero, that is, if βl is an integer multiple of π. However, this is not the case because βl has both a real part and an imaginary part.

$$\beta l = \frac{l}{c\sqrt{2}}\left(\sqrt{\sqrt{\omega^4 + r^2\omega^2} + \omega^2} - i\sqrt{\sqrt{\omega^4 + r^2\omega^2} - \omega^2}\right) \neq m\pi, \quad m = 1, 2, \ldots$$

Only if $r \to 0$ does the imaginary component disappear and resonance occur for $\omega = m\pi c/l$. Therefore, friction prevents resonance from occuring.