Exercise 1

Show that a function which is a power series in the complex variable $x+iy$ must satisfy the Cauchy–Riemann equations and therefore Laplace's equation.

Solution

Suppose that $f(z)$ is a complex-valued function that can be expressed as a power series with complex coefficients c_n.

$$f(z)=\sum_{n=0}^{\infty} c_n z^n$$

The fact that $f(z)$ is complex means that it has a real part and an imaginary part. Since $z=x+iy$, $f(z)$ can be represented as $u(x,y)+iv(x,y)$, where u and v are real functions.

$$u(x,y)+iv(x,y)=\sum_{n=0}^{\infty} c_n(x+iy)^n \tag{1}$$

Differentiate both sides of equation (1) with respect to x.

$$u_x+iv_x=\sum_{n=0}^{\infty} nc_n(x+iy)^{n-1} \tag{2}$$

Differentiate both sides of equation (1) with respect to y.

$$u_y+iv_y=i\sum_{n=0}^{\infty} nc_n(x+iy)^{n-1} \tag{3}$$

Substitute equation (2) into equation (3).

$$\begin{aligned} u_y+iv_y&=i(u_x+iv_x)\\ u_y+iv_y&=iu_x-v_x \end{aligned}$$

Match the real and imaginary parts on both sides to obtain the Cauchy–Riemann equations.

$$\begin{aligned} u_y&=-v_x\\ v_y&=u_x \end{aligned}$$

Differentiate both sides of the first equation with respect to y and both sides of the second equation with respect to x. The mixed partial derivatives, v_{xy} and v_{yx}, are equal by Clairaut's theorem.

$$\left.\begin{aligned} u_{yy}&=-v_{xy}\\ v_{yx}&=u_{xx} \end{aligned}\right\} \quad\to\quad -u_{yy}=u_{xx} \quad\to\quad u_{xx}+u_{yy}=0$$

Therefore, a function which is a power series in the complex variable $x+iy$ must satisfy the Cauchy–Riemann equations and consequently Laplace's equation.